

Navier-Stokes equation and forward-backward stochastic differential system in the Besov spaces

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Abstract

The Navier-Stokes equation on \mathbb{R}^d ($d \geq 3$) formulated on Besov spaces is considered. Using a stochastic forward-backward differential system, the local existence of a unique solution in $B_{p,p}^r$, with $r > 1 + \frac{d}{p}$ is obtained. We also show the convergence to solution of the Euler equation when the viscosity tends to zero. Moreover, we prove the local existence of a unique solution in $B_{p,q}^r$, with $p > 1$, $1 \leq q < \infty$, $r > \max(1, \frac{d}{p})$; here the maximal time interval depends on the viscosity ν .

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1 Introduction

The motion and evolution of the velocity field in an incompressible fluid can be described by the following Navier-Stokes equation in $[0, T] \times \mathbb{R}^d$ ($d \geq 2$),

$$\boxed{\text{e1}} \quad (1.1) \quad \begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p, \\ \nabla \cdot u = 0, \quad u(0) = u_0, \quad t \in [0, T], \end{cases}$$

where $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ represents the velocity field and $\nabla \cdot u$ denotes its divergence, $p : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ denotes the pressure, and $\nu > 0$ is the viscosity. In particular p is a function satisfying the following equation:

$$\boxed{\text{e2}} \quad (1.2) \quad \Delta p(t, x) = - \sum_{i,j=1}^d \partial_i u^j(t, x) \partial_j u^i(t, x), \quad \forall t \in [0, T],$$

where for the vector field $u = (u^1, \dots, u^d)$, $\partial_i u^j$, $1 \leq i, j \leq d$ denotes the partial derivative with respect to the i -th variable for the j -th component of u .

The Navier-Stokes equation (1.1) is and has been for a long time the subject of many works. We refer for example to the books [11], [26], [30] and the references therein. More recently the Navier-Stokes equation has been studied using stochastic methods. In [24] Y. Le Jan and A. S. Sznitman used a branching process in Fourier space to show the local existence and uniqueness of solutions on \mathbb{R}^d . In [12], P. Constantin and G. Iyer obtained a stochastic representation of (1.1) by using the associated stochastic Lagrangian paths; in particular the solution of (1.1) is seen to be equivalent to the solution of a stochastic-functional system. And in [19], G. Iyer derived the local existence of a unique solution in Hölder spaces on the torus by proving the corresponding result for the equivalent stochastic-functional system. Moreover, a backward stochastic Lagrangian path was used by X. C. Zhang in [34] to give a stochastic representation for the backward version of (1.1), and the local existence and uniqueness of the solution in Sobolev space were also obtained based on such representation. In [6], B. Busnello proved the existence and uniqueness of the solution on \mathbb{R}^2 by analyzing the corresponding vorticity equation and the Biot-Savart law. Based on such formulation, in [7] local existence and uniqueness of the solution in Hölder space on \mathbb{R}^3 was shown by B. Busnello, F. Flandoli and M. Romito via a generalized Feynman-Kac formula. In [2] S. Albeverio and Y. Belopolskaya obtained the local existence of a unique solution in Hölder space on \mathbb{R}^d via a semi-group expression for this solution. Moreover, the global existence of the solution on the torus has been studied in [20] and [34]. Recently, in [13], A. B. Cruzeiro and Z. M. Qian proved global existence of a unique solution in Sobolev spaces on the two dimensional torus using the vorticity equation and the associated backward SDE.

On the other hand, in [10], a characterization of the solution for Navier-Stokes equation was derived by F. Cipriano and A. B. Cruzeiro through some stochastic variational

principle, which was formulated on the group of volume preserving diffeomorphisms. We refer to [1] for the generalization of this approach to general Lie groups. In [14], A. B. Cruzeiro and E. Shamarova established an equivalence between the solution of (1.1) and the solution of a forward-backward stochastic differential equation on the space of volume preserving maps.

The purpose of this article is to study the local existence of a unique solution for Navier-Stokes equation in \mathbb{R}^d with $d \geq 3$ in Besov spaces via the forward-backward stochastic differential systems (2.3) and (5.1). Our methods are partially inspired by those of [12], [14] and [13]. More precisely, inspired by [14], we will prove the local existence of a unique solution for (1.1) by proving the corresponding property for an equivalent forward-backward stochastic differential system. As in [12], we use a stochastic Lagrangian path (forward-equation) which is independent of the viscosity ν . Inspired by [13], we can also choose a different forward equation in the stochastic functional system.

A certain linear backward SDE was first introduced by J. M. Bismut in [4]. In [27], E. Pardoux and S. Peng made the important observation that there exists a unique solution for a general (non-linear) backward SDE. In [28], the connection between forward-backward SDEs and quasi-linear PDEs was established by E. Pardoux and S. Peng, which can be viewed as a generalization of Feynman-Kac's formula; see also [25] and the reference therein for an introduction of the forward-backward SDEs with more general forms.

There are some results on the existence of solutions in the Besov spaces for (1.1), most of them were proved via analytic methods. For example, on \mathbb{R}^3 , the existence of a strong solution in (homogeneous Besov space) $\dot{B}_{p,q}^{\frac{3}{p}-1}$ with $1 \leq p < \infty$, $1 \leq q \leq \infty$ was shown in [9], and such result was extended to the case of $p = \infty$ in [3], see also [23]. For the definition of strong solutions, we refer to [17]. On the other hand, in [8], the existence of a unique strong solution in some subspace of $\dot{B}_{p,\infty}^{\frac{3}{p}-1}$ for small initial data on \mathbb{R}^3 for $3 < p \leq 6$ was obtained. In [33], in the following three spaces (or their subspaces): (1) $B_{p,\infty}^r$ with $1 \leq p \leq 2$, $r > 1$, $r > \frac{d}{p} - 1$; (2) $B_{2,1}^r$ with $r > 1$, $r \geq \frac{d}{2} - 1$; (3) $B_{2,q}^r$ with $1 < q < \infty$, $r > 1$, $r > 1 + \frac{d}{2} - \frac{2}{q}$, the existence of a unique strong solution for small initial data was shown.

In fact, in order to get the results above, the viscosity coefficient ν needs to be strictly positive; in our paper we can show the local existence of a unique solution in $B_{p,p}^r$ with $p > 1$, $r > 1 + \frac{d}{p}$, and the maximal time interval is independent of ν , which implies that such result can be applied to the Euler equation. In the proof, we use the same Lagrangian path (forward equation) as the one in [12] (also the same as in [2],[7],[14], [34]). We also show a result about the convergence of the solution as $\nu \rightarrow 0$. More generally, in the spaces $B_{p,q}^r$ with $p > 1$, $1 \leq q < \infty$, $r > \max(1, \frac{d}{p})$ (or in subspaces of these spaces), we also prove local existence of a unique solution. Here we adopt a different Lagrangian path from the one in [14] and, in this case, however, the

maximal time interval for the solution depends on ν .

During the finalization of this paper we found a recent work [15] by F. Delbaen, J. N. Qiu and S. J. Tang, where a forward-backward stochastic functional system different from ours was introduced. Moreover, the local existence of a unique solution of (1.1) in Sobolev space was derived by studying the corresponding property of such system, and the maximal time interval depends on the viscosity ν .

This article is organized as follows: in Section 2 we give a brief description of the framework, and prove some Lemmas that will be needed later; in Section 3 we present the unique local existence of the solution in $B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ with $p > 1$, $r > 1 + \frac{d}{p}$ for (1.1) and in Section 4 we study the limit behaviour of the Navier-Stokes solution as the viscosity ν tends to 0. In Section 5 we prove the unique local existence of the solution in $B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d)$ with $p > 1$, $1 \leq q < \infty$, $r > \max(1, \frac{d}{p})$ for equation (1.1).

2 Notations and preliminary results

Throughout this paper we consider the Navier-Stokes equation in \mathbb{R}^d with $d \geq 3$. Let $C_c^\infty(\mathbb{R}^d)$ denote the set of smooth functions on Euclidean space \mathbb{R}^d which have compact supports and $C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ denote the set of smooth vector fields in \mathbb{R}^d with compact supports. Analogously $C_b^\infty(\mathbb{R}^d)$ stands for the set of smooth bounded functions. For a vector field $v = (v^1, \dots, v^d)$ in \mathbb{R}^d , the divergence of v is denoted by $\nabla \cdot v$, and $\partial_i v^j$, $1 \leq i, j \leq d$ stands for the partial derivative with respect to the i -th variable of the j -th component of v . If \mathbb{N} is the set of natural numbers, for every non-negative $k \in \mathbb{N}$ and every real number $p > 1$, the Sobolev space $W^{k,p}(\mathbb{R}^d; \mathbb{R}^d)$ of vector fields is the completion of $C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ under the following norm,

$$\|v\|_{W^{k,p}} := \sum_{i=0}^k \|\nabla^i v\|_{L^p},$$

where ∇^i is the i -th differential of v , and $\|\cdot\|_{L^p}$ denotes the L^p norm (with respect to Lebesgue measure). The Sobolev space $W^{k,p}(\mathbb{R}^d)$ of functions can be defined in the same way.

As in [29] and [31, Section 1.2], we introduce the Besov space $B_{p,q}^{k+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$ in \mathbb{R}^d , where k is a non-negative integer, $0 < \alpha < 1$, $1 < p < \infty$, $1 \leq q \leq \infty$.

If $1 \leq q < \infty$,

$$B_{p,q}^{k+\alpha}(\mathbb{R}^d; \mathbb{R}^d) := \left\{ v \in W^{k,p}(\mathbb{R}^d; \mathbb{R}^d), \int_{\mathbb{R}^d} \frac{\|\nabla^k v(\cdot + y) - \nabla^k v(\cdot)\|_{L^p}^q}{|y|^{d+\alpha q}} dy < \infty \right\}.$$

If $q = \infty$,

$$B_{p,\infty}^{k+\alpha}(\mathbb{R}^d; \mathbb{R}^d) := \left\{ v \in W^{k,p}(\mathbb{R}^d; \mathbb{R}^d), \sup_{|y|>0} \left\{ \frac{\|\nabla^k v(\cdot + y) - \nabla^k v(\cdot)\|_{L^p}}{|y|^\alpha} \right\} < \infty \right\}.$$

For each $v \in B_{p,q}^{k+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$, we define the following quasi-norm

$$\|v\|_{B_{p,q}^{k+\alpha}} := \|v\|_{W^{k,p}} + [v]_{B_{p,q}^{k+\alpha}},$$

where

$$\boxed{\text{e0}} \quad (2.1) \quad [v]_{B_{p,q}^{k+\alpha}} := \left(\int_{\mathbb{R}^d} \frac{\|\nabla^k v(\cdot + y) - \nabla^k v(\cdot)\|_{L^p}^q}{|y|^{d+\alpha q}} dy \right)^{\frac{1}{q}}$$

if $1 \leq q < \infty$, and

$$\boxed{\text{e0a}} \quad (2.2) \quad [v]_{B_{p,\infty}^{k+\alpha}} := \sup_{|y|>0} \left\{ \frac{\|\nabla^k v(\cdot + y) - \nabla^k v(\cdot)\|_{L^p}}{|y|^\alpha} \right\}$$

if $q = \infty$.

To see the various equivalent definitions of $B_{p,q}^{k+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$, (for example via the Fourier multiplication) one can refer to [31, Chapter 1] or [32, Section 2.5]. In particular, $B_{p,p}^{k+\alpha}(\mathbb{R}^d)$ is the fractional order Sobolev space $W^{k+\alpha,p}(\mathbb{R}^d)$, see [31, Chapter 1].

Remark 2.1. Note that the space $B_{p,p}^r(\mathbb{R}^d)$ (or $B_{p,q}^r(\mathbb{R}^d)$) may not coincide with $W^{r,p}(\mathbb{R}^d)$ if r is an integer (see [32]). For this reason and unless particularly clarified, we shall consider that r is not an integer in this paper.

Suppose that u is a solution of (1.1) in the time interval $t \in [0, T]$, which is regular enough (for example $u \in C([0, T]; C_b^\infty(\mathbb{R}^d, \mathbb{R}^d))$). Fix a Brownian motion B_t on \mathbb{R}^d , for $0 \leq t \leq s \leq T$ and let $X_s^t(x)$ be the unique solution of the following SDE,

$$\begin{cases} dX_s^t(x) = \sqrt{2\nu} dB_s - u(T-s, X_s^t(x)) ds \\ X_t^t(x) = x, \end{cases}$$

We define $Y_s^t(x) := u(T-s, X_s^t(x))$, $Z_s^t(x) := \nabla u(T-s, X_s^t(x))$. Applying Itô's formula directly, we derive the following (forward-backward) stochastic differential system on function space whose solution is $(X_s^t(x), Y_s^t(x), Z_s^t(x), u(t, x), p(t, x))$,

$$\boxed{\text{e3}} \quad (2.3) \quad \begin{cases} dX_s^t(x) = \sqrt{2\nu} dB_s - u(T-s, X_s^t(x)) ds \\ dY_s^t(x) = \sqrt{2\nu} Z_s^t(x) dB_s + \nabla p(T-s, X_s^t(x)) ds \\ Y_t^t(x) = u(T-t, x), \quad \Delta p(t, x) = -\sum_{i,j=1}^3 \partial_i u^j(t, x) \partial_j u^i(t, x) \\ X_t^t(x) = x, Y_T^t(x) = u_0(X_T^t(x)). \end{cases}$$

On the other hand, if $(X_s^t(x), Y_s^t(x), Z_s^t(x), u(t, x), p(t, x))$ is a solution of (2.3) and u is regular enough, for example, $u \in C([0, T]; C_b^3(\mathbb{R}^d; \mathbb{R}^d))$, then $(X_s^t(x), Y_s^t(x), Z_s^t(x))$ satisfies a backward SDE where u and p appear in the coefficients, so by [28, Theorem

3.2], the vector field $u(t, x) := Y_{T-t}^{T-t}(x)$ satisfies equation (1.1) for $t \in [0, T]$. In particular, we can show that $\nabla \cdot u(t, x) = 0$ due to the expression of $p(t, x)$ in (2.3). We refer to [14] for the formulation of such system on the space of diffeomorphisms group. Inspired by this argument, we will construct a solution of the system (2.3) in Besov space, which is still a solution for Navier-Stokes equation (1.1) in such function space.

Let us introduce the following notation:

$$\begin{aligned} F_v &:= \nabla N G_v, \quad G_v := \sum_{i,j=1}^d \partial_i v^j \partial_j v^i, \\ Nf(x) &:= C(d) \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-2}} dy, \quad \forall f \in C_c^\infty(\mathbb{R}^d), \end{aligned} \tag{e5a} \quad (2.4)$$

where N is the Newton's potential in \mathbb{R}^d , $C(d)$ is a constant depending on d , and $\Delta Nf(x) = f(x)$ for every $f \in C_c^\infty(\mathbb{R}^d)$. Furthermore potential theory (see [29]) assures that Nf is well defined for every $f \in L^{p'}(\mathbb{R}^d)$ with $1 < p' < \frac{d}{2}$.

Given a $u_0 \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$, $v \in C([0, T]; C_c^\infty(\mathbb{R}^d; \mathbb{R}^d))$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot v(t) = 0$ for every t , let (X_s^t, Y_s^t, Z_s^t) (we omit the index v here) be the unique solution of the following BSDE,

$$\begin{cases} dX_s^t(x) = \sqrt{2\nu} dB_s - v(T-s, X_s^t(x)) ds \\ dY_s^t(x) = \sqrt{2\nu} Z_s^t(x) dB_s - F_v(T-s, X_s^t(x)) ds \\ X_t^t(x) = x, Y_T^t(x) = u_0(X_T^t(x)), \end{cases} \tag{e5} \quad (2.5)$$

where

$$F_v(t, x) := F_{v(t)}(x) = \nabla N G_{v(t)}(x), \quad t \in [0, T], x \in \mathbb{R}^d$$

for simplicity.

By the potential theory, F_v is well defined for every $v \in W^{2,p'}(\mathbb{R}^d) \cap W^{2,p}(\mathbb{R}^d)$ with some $1 < p' < \frac{d}{2}$ and $p > d$. Following the methods in [26, Chapter 2], see also [29], we can prove the following Lemma on the Besov norm bounds of F_v .

In this paper, the constant C may change in different lines in the proofs, but will be independent of the variables stated in the conclusion.

[11] Lemma 2.2. *Let $d < p < \infty$, $1 \leq q \leq \infty$ and $1 < p' < \frac{d}{2}$. Then for all $v \in \bigcap_{r>1} B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d) \cap W^{2,p'}(\mathbb{R}^d; \mathbb{R}^d)$ satisfying $\nabla \cdot v = 0$, and for every $0 < \alpha < 1$, we have*

$$\begin{aligned} \|F_v\|_{L^p} &\leq C_1 \|\nabla v\|_{L^\infty} \|v\|_{L^p}, \\ \|F_v\|_{B_{p,q}^{1+\alpha}} &\leq C_1 \|\nabla v\|_{L^\infty} \|v\|_{B_{p,q}^{1+\alpha}}, \\ \|F_v\|_{B_{p,q}^{2+\alpha}} &\leq C_1 \|v\|_{W^{2,p}} \|v\|_{B_{p,q}^{2+\alpha}}, \end{aligned} \tag{e6} \quad (2.6)$$

where C_1 is a positive constant independent of v and p' .

Proof. Since $\nabla \cdot v = 0$,

$$\boxed{\text{e6a}} \quad (2.7) \quad G_v(x) = \sum_{i,j} \partial_i v^j(x) \partial_j v^i(x) = \sum_i \partial_i \left(\sum_j v^j(x) \partial_j v^i(x) \right) := \sum_i \partial_i f_i(x),$$

hence $F_v(x) = \nabla N G_v(x) = \sum_i \nabla N \partial_i f_i(x)$. It is easy to check that $\nabla N \partial_i$ is a singular integral operator as defined in [29, Chapter 2], so it is bounded in L^p space for every $1 < p < \infty$ (see [29, Theorem 3, Chapter 2]), which implies that

$$\|F_v\|_{L^p} \leq C \sum_i \|f_i\|_{L^p} \leq C \|\nabla v\|_{L^\infty} \|v\|_{L^p}.$$

Note that $\nabla F_v(x) = \nabla^2 N G_v(x)$ and that

$\nabla^2 N$ is a singular integral operator, we obtain,

$$\|\nabla F_v\|_{L^p} \leq C \|G_v\|_{L^p} \leq C \|\nabla v\|_{L^\infty} \|\nabla v\|_{L^p}.$$

In the same way as above we can show that,

$$\|\nabla^2 F_v\|_{L^p} \leq C \|\nabla G_v\|_{L^p} \leq C \|\nabla v\|_{L^\infty} \|\nabla^2 v\|_{L^p}.$$

For every $y \in \mathbb{R}^d$, let $\tilde{G}_v^y(x) := G_v(x+y) - G_v(x)$. Then

$$\begin{aligned} & \|\nabla \tilde{G}_v^y\|_{L^p} \\ & \leq C \left(\|\nabla v\|_{L^\infty} \|\nabla^2 v(\cdot+y) - \nabla^2 v(\cdot)\|_{L^p} + \|\nabla^2 v\|_{L^p} \|\nabla v(\cdot+y) - \nabla v(\cdot)\|_{L^\infty} \right). \end{aligned}$$

Since N is translation invariant,

$$\nabla^2 F_v(x+y) - \nabla^2 F_v(x) = \nabla^2 N(\nabla \tilde{G}_v^y)(x).$$

Applying singular integral estimates to $\nabla^2 N$ we obtain

$$\boxed{\text{e7aa}} \quad (2.8) \quad \begin{aligned} & \|\nabla^2 F_v(\cdot+y) - \nabla^2 F_v(\cdot)\|_{L^p} \leq C \|\nabla \tilde{G}_v^y\|_{L^p} \\ & \leq C \left(\|\nabla v\|_{L^\infty} \|\nabla^2 v(\cdot+y) - \nabla^2 v(\cdot)\|_{L^p} + \|\nabla^2 v\|_{L^p} \|\nabla v(\cdot+y) - \nabla v(\cdot)\|_{L^\infty} \right). \end{aligned}$$

According to the embedding theorem [32, Theorem 2.8.1], since $v \in B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$ and $p > d$,

$$\boxed{\text{e7d}} \quad (2.9) \quad |\nabla v(x) - \nabla v(y)| \leq C \|v\|_{B_{p,q}^{2+\alpha}} |x-y|^{r(p)}, \quad \forall x, y \in \mathbb{R}^d,$$

$$\|\nabla v\|_{L^\infty} \leq C \|v\|_{W^{2,p}} \leq C \|v\|_{B_{p,q}^{2+\alpha}},$$

where $r(p)$ is given by

$$\boxed{\text{e7a}} \quad (2.10) \quad r(p) := \begin{cases} \min\{1 + \alpha - \frac{d}{p}, 1\}, & \text{if } 1 + \alpha - \frac{d}{p} \neq 1, \\ \text{any real number } \in (\alpha, 1), & \text{if } 1 + \alpha - \frac{d}{p} = 1. \end{cases}$$

Applying this to (2.8), we deduce that

$$\begin{aligned} \|\nabla^2 F_v(\cdot + y) - \nabla^2 F_v(\cdot)\|_{L^p} &\leq C \left(\|v\|_{W^{2,p}} \|\nabla^2 v(\cdot + y) - \nabla^2 v(\cdot)\|_{L^p} \right. \\ &\quad \left. + \|v\|_{W^{2,p}} \|v(t)\|_{B_{p,q}^{2+\alpha}} \left(|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}} \right) \right), \end{aligned}$$

which together with (2.1) and (2.2), we conclude that

$$\|F_v\|_{B_{p,q}^{2+\alpha}} \leq C \|v\|_{W^{2,p}} \|v\|_{B_{p,q}^{2+\alpha}}.$$

Combining the above estimates together, we may obtain the third estimate in (2.6).

The second estimate in (2.6) may be proved following a similar procedure. \square

$\boxed{\text{r2.1}}$ **Remark 2.3.** According to potential theory, in general F_v is not well defined without the $L^{p'}$ integrable condition on v for some $1 < p' < \frac{d}{2}$. Note that (2.6) is independent of p' and the $L^{p'}$ norm, although we assume $v \in W^{2,p'}(\mathbb{R}^d; \mathbb{R}^d)$ to ensure that the Newton's potential N is well defined. By an approximation argument (see the proof of Proposition 3.10 below), if $\nabla \cdot v = 0$, and $v \in W^{2,p}(\mathbb{R}^d; \mathbb{R}^d)$ with some $p > d$, F_v is still well defined.

From now on, in this paper, for $1 < p < \infty$, $1 \leq q \leq \infty$, $1 < p' < \frac{d}{2}$, we define

$$\boxed{\text{e2a}} \quad (2.11) \quad \begin{aligned} \mathcal{S}(p, q, p', T) &:= \left\{ v \in C([0, T]; C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)); \right. \\ &\quad \left. \sup_{t \in [0, T]} (\|v(t)\|_{B_{p,q}^r} + \|v(t)\|_{W^{2,p'}}) < \infty, \text{ for } \forall r > 1; \quad \nabla \cdot v(t) = 0, \forall t \in [0, T] \right\}. \end{aligned}$$

For every $v \in \mathcal{S}(p, q, p', T)$ with some $1 < p < \infty$, $1 \leq q \leq \infty$, $1 < p' < \frac{d}{2}$, $u_0 \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$ with $\nabla \cdot u_0 = 0$, F_v is well defined and we can define $\tilde{\mathcal{J}}_\nu(u_0, v) \in C([0, T]; C_b^\infty(\mathbb{R}^d; \mathbb{R}^d))$ such that

$$\tilde{\mathcal{J}}_\nu(u_0, v)(t) := \mathbf{P}(Y_{T-t}^{T-t}(.))$$

for every $t \in [0, T]$, where \mathbf{P} denotes the Leray-Hodge projection on the space of divergence free vector fields. In particular, for every $v \in \mathcal{S}(p, q, p', T)$, we define

$$\mathcal{J}_\nu(v) := \tilde{\mathcal{J}}_\nu(v(0), v).$$

Intuitively, if we can find a fixed point v for the map \mathcal{J}_ν (in some function space), then v will be a solution of (2.3), hence a solution of (1.1). In this work we will prove

that we can extend such map \mathcal{J}_ν to be defined in some Besov space, that \mathcal{J}_ν has a unique fixed point in such space, and that the fixed point v can be viewed as a solution of the Navier-Stokes equation (1.1).

We shall need the following result:

13 **Lemma 2.4.** *Suppose that $v_m \in \mathcal{S}(p, q, p', T)$, $m = 1, 2$, for some $d < p < \infty$, $1 \leq q \leq \infty$, $1 < p' < \frac{d}{2}$, $T > 0$. Then for every $0 < \alpha < 1$, $t \in [0, T]$, the functions $F_{v_m(t)}, G_{v_m(t)}$ defined by (2.4) satisfy the following inequalities,*

e8aa (2.12)
$$\begin{aligned} \|F_{v_1(t)} - F_{v_2(t)}\|_{W^{1,p}} &\leq C_1 \sup_{m=1,2} \|v_m(t)\|_{W^{2,p}} \|v_1(t) - v_2(t)\|_{W^{1,p}}, \\ \|F_{v_1(t)} - F_{v_2(t)}\|_{B_{p,q}^{1+\alpha}} &\leq C_1 K \|v_1(t) - v_2(t)\|_{B_{p,q}^{1+\alpha}}, \end{aligned}$$

where $K := \sup_{t \in [0, T], m=1,2} \|v_m(t)\|_{B_{p,q}^{2+\alpha}}$, C_1 is independent of v , K , p' and T .

Proof. It is a consequence of Lemma 2.1, the bilinearity property of the map $v \rightarrow F_v$ and the inequality,

e22a (2.13)
$$\begin{aligned} &\left| f_1(x)g_1(x) - f_2(x)g_2(x) - (f_1(y)g_1(y) - f_2(y)g_2(y)) \right| \\ &\leq |g_1(x)| |f_1(x) - f_2(x) - (f_1(y) - f_2(y))| + |f_1(y)| |g_1(x) - g_2(x) - (g_1(y) - g_2(y))| \\ &\quad + |f_2(x) - f_2(y)| |g_1(x) - g_2(x)| + |g_2(x) - g_2(y)| |f_1(y) - f_2(y)| \end{aligned}$$

Indeed we can write $F_{v_1(t)} - F_{v_2(t)} = \sum_i \nabla N \left(\partial_i (f_{i,1}(t) - f_{i,2}(t)) \right)$, where $f_{i,m}(t, x) := \sum_j v_m^j(t, x) \partial_j v_m^i(t, x)$. We have,

$$\begin{aligned} \|F_{v_1(t)} - F_{v_2(t)}\|_{L^p} &\leq C \sum_i \|f_{i,1}(t) - f_{i,2}(t)\|_{L^p} \\ &\leq C \left(\|\nabla v_1(t)\|_{L^\infty} \|v_1(t) - v_2(t)\|_{L^p} + \|v_2(t)\|_{L^\infty} \|\nabla v_1(t) - \nabla v_2(t)\|_{L^p} \right) \\ &\leq C \sup_{m=1,2} \|v_m(t)\|_{W^{2,p}} \|\nabla v_1(t) - \nabla v_2(t)\|_{W^{1,p}}, \end{aligned}$$

As $\nabla(F_{v_1(t)} - F_{v_2(t)}) = \nabla^2 N(G_{v_1(t)} - G_{v_2(t)})$, we have,

$$\begin{aligned} \|\nabla F_{v_1(t)} - \nabla F_{v_2(t)}\|_{L^p} &\leq C \|G_{v_1(t)} - G_{v_2(t)}\|_{L^p} \\ &\leq C (\|\nabla v_1(t)\|_{L^\infty} + \|\nabla v_2(t)\|_{L^\infty}) \|\nabla v_1(t) - \nabla v_2(t)\|_{L^p} \\ &\leq C \sup_{m=1,2} \|v_m(t)\|_{W^{2,p}} \|\nabla v_1(t) - \nabla v_2(t)\|_{W^{1,p}}. \end{aligned}$$

therefore the first estimate holds.

Concerning the second estimate, writting $\tilde{G}_{v_m(t)}^y(x) := G_{v_m(t)}(x+y) - G_{v_m(t)}(x)$, $m = 1, 2$, by (2.13) we can get,

e23aa

$$(2.14) \quad \begin{aligned} & |\tilde{G}_{v_1(t)}^y(x) - \tilde{G}_{v_2(t)}^y(x)| \\ & \leq C \left(\|\nabla v_1(t)\|_{L^\infty} |\nabla v_1(t, x+y) - \nabla v_1(t, x) - (\nabla v_2(t, x+y) - \nabla v_2(t, x))| \right. \\ & \quad \left. + |\nabla v_2(t, x+y) - \nabla v_2(t, x)| \left(|\nabla v_1(t, x+y) - \nabla v_2(t, x+y)| + |\nabla v_1(t, x) - \nabla v_2(t, x)| \right) \right). \end{aligned}$$

Then according to (2.9) we derive,

$$\begin{aligned} & \|(\nabla F_{v_1(t)}(\cdot+y) - \nabla F_{v_1(t)}(\cdot)) - (\nabla F_{v_2(t)}(\cdot+y) - \nabla F_{v_2(t)}(\cdot))\|_{L^p} = \|\nabla^2 N(\tilde{G}_{v_1(t)}^y - \tilde{G}_{v_2(t)}^y)\|_{L^p} \\ & \leq C \|\tilde{G}_{v_1(t)}^y - \tilde{G}_{v_2(t)}^y\|_{L^p} \leq CK \|(\nabla v_1 - \nabla v_2)(t, \cdot+y) - (\nabla v_1 - \nabla v_2)(t, \cdot)\|_{L^p} \\ & + CK (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) \|\nabla v_1(t) - \nabla v_2(t)\|_{L^p}, \end{aligned}$$

which implies that

$$[\nabla F_{v_1(t)} - \nabla F_{v_2(t)}]_{B_{p,q}^{1+\alpha}} \leq CK \|v_1(t) - v_2(t)\|_{B_{p,q}^{1+\alpha}}.$$

□

3 The local existence theorem in $B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$

We first prove the following estimate.

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Lemma 3.1. *Suppose that $v \in \mathcal{S}(p, p, p', T)$, where $d < p < \infty$, $1 < p' < \frac{d}{2}$, $T > 0$. Let X be the unique solution of the first equation of (2.5) (with coefficients v). Then for every $0 < \alpha < 1$ and $f \in B_{p,p}^{2+\alpha}(\mathbb{R}^d)$, we have*

e24a

$$(3.1) \quad \sup_{0 \leq s \leq t \leq T} \|f \circ X_s^t\|_{B_{p,p}^{2+\alpha}} \leq C_1 e^{C_1 K T} (1 + T^2 K^2) \|f\|_{B_{p,p}^{2+\alpha}}, \text{ a.s.},$$

where $f \circ X_s^t$ denotes the composition of function f and the map $X_s^t : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $K := \sup_{t \in [0, T]} \|v(t)\|_{B_{p,p}^{2+\alpha}}$, C_1 is a positive constant independent of K , ν , T , p' , v and f .

Proof. Step 1: We need the $W^{2,p}$ estimate in [34, Lemma 3.5] which is standard, but for the reader's convenience, we follow the same method of that reference and write the proof again.

Since $p > d$, by the Sobolev embedding theorem,

$$(\|v(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty}) \leq C \|v(t)\|_{W^{2,p}} \leq C \|v(t)\|_{B_{p,p}^{2+\alpha}} \leq CK.$$

As $v \in \mathcal{S}(p, p, p', T)$, there is a version of $X_s^t(\cdot)$ which is C^∞ differentiable and its first and second order differentials $\nabla X_s^t, \nabla^2 X_s^t$ ($0 \leq t \leq s \leq T$) satisfy the following equation,

$$\boxed{\text{e7}} \quad (3.2) \quad \begin{cases} d\nabla X_s^t(x) = -\nabla v(T-s, X_s^t(x)) \nabla X_s^t(x) ds \\ d\nabla^2 X_s^t(x) = -\nabla v(T-s, X_s^t(x)) \nabla^2 X_s^t(x) ds - \nabla^2 v(T-s, X_s^t(x)) (\nabla X_s^t(x))^2 ds \\ \nabla X_s^t(x) = \mathbf{I}, \quad \nabla^2 X_s^t(x) = 0, \end{cases}$$

where \mathbf{I} denotes identity map in \mathbb{R}^d . Since $\nabla \cdot v(t) = 0$, $X_s^t(\cdot)$ and $(X_s^t)^{-1}(\cdot)$ are volume preserving maps (see [22]), for every $h \in L^1(\mathbb{R}^d)$, $0 \leq t \leq s \leq T$, $h \circ X_s^t(\cdot)$ is almost surely a.e. well defined (with respect to Lebesgue measure), and

$$\boxed{\text{e10aa}} \quad (3.3) \quad \int_{\mathbb{R}^d} h(X_s^t(x)) dx = \int_{\mathbb{R}^d} h(x) dx, \quad a.s..$$

So we have, for $f \geq 0$,

$$\boxed{17.1} \quad (3.4) \quad \int_{\mathbb{R}^d} f^p(X_s^t(x)) dx = \int_{\mathbb{R}^d} f^p(x) dx = \|f\|_{L^p}^p, \quad a.s..$$

Note that $\|\nabla v(t)\|_{L^\infty} \leq CK$ and the martingale part of (3.2) vanishes. Applying Grownwall lemma, for every $0 \leq t \leq s \leq T$, we have

$$\boxed{\text{e8}} \quad (3.5) \quad |\nabla X_s^t(x)| \leq Ce^{CKT}, \quad a.s., \quad \forall x \in \mathbb{R}^d.$$

Hence from (3.2),

$$|\nabla^2 X_s^t(x)| \leq K \int_t^s |\nabla^2 X_r^t(x)| dr + Ce^{CKT} \int_t^s |\nabla^2 v(T-r, X_r^t(x))| dr,$$

then by Grownwall lemma, we derive that

$$|\nabla^2 X_s^t(x)| \leq Ce^{CKT} \int_t^s |\nabla^2 v(T-r, X_r^t(x))| dr,$$

together with (3.3) and Hölder's inequality,

$$\boxed{\text{e9aa}} \quad (3.6) \quad \begin{aligned} \int_{\mathbb{R}^d} |\nabla^2 X_s^t(x)|^p dx &\leq CT^{p-1} e^{CKT} \int_t^s \left(\int_{\mathbb{R}^d} |\nabla^2 v(T-r, X_r^t(x))|^p dx \right) dr \\ &\leq CT^p e^{CKT} \sup_{t \in [0, T]} \|v(t)\|_{W^{2,p}}^p. \end{aligned}$$

Since $f \in B_{p,p}^{2+\alpha}(\mathbb{R}^d)$, $f \in C^1(\mathbb{R}^d)$ due to Sobolev embedding theorem, so $\nabla(f \circ X_s^t)(x) = \nabla f(X_s^t(x)) \nabla X_s^t(x)$, by (3.3) and (3.5), we deduce that

$$\boxed{17.2} \quad (3.7) \quad \int_{\mathbb{R}^d} |\nabla(f \circ X_s^t)(x)|^p dx \leq Ce^{CKT} \|\nabla f\|_{L^p}^p, \quad a.s..$$

Again note that if $f \in C^2(\mathbb{R}^d) \cap B_{p,p}^{2+\alpha}(\mathbb{R}^d)$,

$$\boxed{\text{e10a}} \quad (3.8) \quad \nabla^2(f \circ X_s^t)(x) = \nabla^2 f(X_s^t(x))(\nabla X_s^t(x))^2 + \nabla f(X_s^t(x))\nabla^2 X_s^t(x),$$

by (3.3) (3.5) and (3.6),

$$\begin{aligned} \boxed{\text{17.3}} \quad (3.9) \quad & \int_{\mathbb{R}^d} |\nabla^2(f \circ X_s^t)(x)|^p dx \\ & \leq C \|\nabla X_s^t(\cdot)\|_{L^\infty}^{2p} \int_{\mathbb{R}^d} |\nabla^2 f(X_s^t(x))|^p dx + C \|\nabla f\|_{L^\infty}^p \int_{\mathbb{R}^d} |\nabla^2 X_s^t(x)|^p dx \\ & \leq C e^{CKT} \|\nabla^2 f\|_{L^p}^p + CT^p K^p e^{CKT} \|\nabla f\|_{W^{1,p}}^p \leq C(1 + T^p K^p) e^{CKT} \|f\|_{W^{2,p}}^p, \end{aligned}$$

where the second inequality follows from the Sobolev embedding theorem which in particular implies that $\|\nabla f\|_{L^\infty} \leq C \|\nabla f\|_{W^{1,p}}$.

For general $f \in B_{p,p}^{2+\alpha}(\mathbb{R}^d)$, we choose a sequence $\{f_n\}_{n=1}^\infty \subseteq C^2(\mathbb{R}^d) \cap B_{p,p}^{2+\alpha}(\mathbb{R}^d)$, such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{W^{2,p}} = 0$, by (3.3), (3.8) and approximation procedure, we know $f \circ X_s^t \in W^{2,p}(\mathbb{R}^d)$, and the estimate (3.8), (3.9) still hold.

Step 2: By (3.5), for every $0 \leq t \leq s \leq T$, $x, y \in \mathbb{R}^d$,

$$\boxed{\text{e10}} \quad (3.10) \quad |X_s^t(x) - X_s^t(y)| \leq C e^{CKT} |x - y|, \text{ a.s..}$$

Let $\Gamma_s^t(x, y) := \nabla X_s^t(x) - \nabla X_s^t(y)$. By (3.2) and noting that the martingale part vanishes, we get that

$$\begin{aligned} \boxed{\text{e11}} \quad (3.11) \quad & |\Gamma_s^t(x, y)| \leq C \int_t^s \left((|\nabla v(T-r, X_r^t(x))| |\Gamma_r^t(x, y)|) \right. \\ & \left. + (|\nabla v(T-r, X_r^t(x)) - \nabla v(T-r, X_r^t(y))| |\nabla X_r^t(y)|) \right) dr. \end{aligned}$$

Together with (2.9), we deduce that

$$|\nabla v(t, x) - \nabla v(t, y)| \leq C \|v(t)\|_{B_{p,p}^{2+\alpha}} |x - y|^{r(p)}, \quad \forall x, y \in \mathbb{R}^d,$$

where $r(p)$ is defined by (2.10). Hence for every $x, y \in \mathbb{R}^d$,

$$\boxed{\text{e13}} \quad (3.12) \quad |\nabla v(T-r, X_r^t(x)) - \nabla v(T-r, X_r^t(y))| \leq CK |X_r^t(x) - X_r^t(y)|^{r(p)} \leq CK e^{CKT} |x - y|^{r(p)}, \text{ a.s.,}$$

where we have used the estimate (3.10). Applying such estimate to (3.11), we obtain

$$|\Gamma_s^t(x, y)| \leq CK \int_t^s |\Gamma_r^t(x, y)| dr + CT K e^{CKT} |x - y|^{r(p)}.$$

Finally, by Grownwall lemma, for every $0 \leq t \leq s \leq T$,

$$\boxed{\text{e28}} \quad (3.13) \quad |\nabla X_s^t(x) - \nabla X_s^t(y)| = |\Gamma_s^t(x, y)| \leq CT K e^{CKT} |x - y|^{r(p)}, \text{ a.s..}$$

Step 3: Note that for every $h \in B_{p,p}^\alpha(\mathbb{R}^d)$,

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|h(X_s^t(x+y)) - h(X_s^t(x))|^p}{|y|^{d+\alpha p}} dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|h(X_s^t(y)) - h(X_s^t(x))|^p}{|x-y|^{d+\alpha p}} dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|h(y) - h(x)|^p}{|(X_s^t)^{-1}(x) - (X_s^t)^{-1}(y)|^{d+\alpha p}} dx dy,
\end{aligned}
\tag{3.14}$$

where in the last step we have used (3.3) and the change of variable. On the other hand, according to (3.10),

$$|x-y| = |X_s^t((X_s^t)^{-1}(x)) - X_s^t((X_s^t)^{-1}(y))| \leq C e^{CKT} |(X_s^t)^{-1}(x) - (X_s^t)^{-1}(y)|,$$

which combining with (3.14) yields that

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|h(X_s^t(x+y)) - h(X_s^t(x))|^p}{|y|^{d+\alpha p}} dx dy \\
& \leq C e^{CKT} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|h(y) - h(x)|^p}{|x-y|^{d+\alpha p}} dx dy = C e^{CKT} [h]_{B_{p,p}^\alpha}^p.
\end{aligned}
\tag{3.15}$$

Step 4: Let $\Upsilon_s^t(x, y) := \nabla^2 X_s^t(x+y) - \nabla^2 X_s^t(x)$. By using similar arguments as above, together with an application of Itô's formula to $\nabla^2 X_s^t(x+y) - \nabla^2 X_s^t(x)$, and using the estimates (3.5), (3.12) and (3.13), we obtain

$$\begin{aligned}
|\Upsilon_s^t(x, y)| &\leq K \int_t^s |\Upsilon_r^t(x, y)| dr + C K e^{CKT} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) \int_t^s |\nabla^2 X_r^t(x)| dr \\
&+ C e^{CKT} \int_t^s |\nabla^2 v(T-r, X_r^t(x+y)) - \nabla^2 v(T-r, X_r^t(x))| dr \\
&+ C T K e^{CKT} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) \int_t^s |\nabla^2 v(T-r, X_r^t(x))| dr, \text{ a.s.},
\end{aligned}$$

Applying Grownwall lemma, Hölder inequality and properties (3.3), (3.15), we have,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Upsilon_s^t(x, y)|^p}{|y|^{d+\alpha p}} dy dx \leq C e^{CKT} (T^p K^p + T^{2p} K^{2p}).
\tag{3.16}$$

Step 5: By (3.5) and (3.8), for every $0 \leq t \leq s \leq T$,

$$\begin{aligned}
[\nabla^2(f \circ X_s^t)]_{B_{p,p}^\alpha}^p &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla^2(f \circ X_s^t)(x+y) - \nabla^2(f \circ X_s^t)(x)|^p}{|y|^{d+\alpha p}} dy dx \\
&\leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|\nabla f\|_{L^\infty}^p \frac{|\Upsilon_s^t(x,y)|^p}{|y|^{d+\alpha p}} dy dx \\
&+ C \left(\int_{\mathbb{R}^d} |\nabla^2 X_s^t(x)|^p dx \right) \left(\int_{\mathbb{R}^d} \frac{\|\nabla f \circ X_s^t(\cdot+y) - \nabla f \circ X_s^t(\cdot)\|_{L^\infty}^p}{|y|^{d+\alpha p}} dy \right) \\
&+ C e^{CKT} \left(\int_{\mathbb{R}^d} |\nabla^2 f \circ X_s^t(x)|^p dx \right) \left(\int_{\mathbb{R}^d} \frac{\|\nabla X_s^t(\cdot+y) - \nabla X_s^t(\cdot)\|_{L^\infty}^p}{|y|^{d+\alpha p}} dy \right) \\
&+ C e^{CKT} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla^2 f \circ X_s^t(x+y) - \nabla^2 f \circ X_s^t(x)|^p}{|y|^{d+\alpha p}} dy dx \\
&:= \sum_{i=1}^4 I_i.
\end{aligned}$$

Due to (3.16) and Sobolev embedding theorem, we have

$$I_1 \leq C e^{CKT} (T^p K^p + T^{2p} K^{2p}) \|f\|_{B_{p,p}^{2+\alpha}}^p.$$

By (2.9) and (3.10), for every $y \in \mathbb{R}^d$,

$$\|\nabla f \circ X_s^t(\cdot+y) - \nabla f \circ X_s^t(\cdot)\|_{L^\infty} \leq C e^{CKT} \|f\|_{B_{p,p}^{2+\alpha}} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}});$$

combining this with (3.6), we get $I_2 \leq CT^p K^p e^{CKT} \|f\|_{B_{p,p}^{2+\alpha}}^p$. According to (3.5) and (3.13), $I_3 \leq CT^p K^p e^{CKT} \|f\|_{B_{p,p}^{2+\alpha}}^p$. By (3.15), $I_4 \leq C e^{CKT} \|f\|_{B_{p,p}^{2+\alpha}}^p$. Putting these estimates together we may conclude that

$$\boxed{17.4} \quad (3.17) \quad [\nabla^2(f \circ X_s^t)]_{B_{p,p}^\alpha}^p \leq C e^{CKT} (1 + T^p K^p + T^{2p} K^{2p}) \|f\|_{B_{p,p}^{2+\alpha}}^p.$$

Since $2TK \leq 1 + T^2 K^2$, estimates (3.4), (3.7), (3.9) and (3.17) imply (3.1). \square

Remark 3.2. If $p \neq q$, estimate (3.14) is no longer useful, we can not get the corresponding estimate (3.1) in space $B_{p,q}^{2+\alpha}$ by the same method in Lemma 3.1.

Now we prove the $B_{p,p}^{2+\alpha}$ bounds for the regular solution of (2.5).

$\boxed{18}$ **Lemma 3.3.** Suppose that $v \in \mathcal{S}(p, p, p', T)$ where $d < p < \infty$, $1 < p' < \frac{d}{2}$, $T > 0$. Let (X, Y, Z) be the unique solution of (2.5) with coefficient v and initial condition $u_0 := v(0)$, and let $g(t, x) := Y_t^t(x)$. Then for any $0 < \alpha < 1$,

$$\boxed{e24d} \quad (3.18) \quad \sup_{t \in [0, T]} \|g(t)\|_{B_{p,p}^{2+\alpha}} \leq C_1 e^{C_1 KT} \|u_0\|_{B_{p,p}^{2+\alpha}} (1 + T^2 K^2) + C_1 T K^2 (1 + T^2 K^2),$$

where $K := \sup_{t \in [0, T]} \|v(t)\|_{B_{p,p}^{2+\alpha}}$, and C_1 is a constant independent of K , ν , T , p' and v .

Proof. Since $v \in \mathcal{S}(p, p, p', T)$, $F_v(t) \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$ by the standard theorem on the regularity for the solution of elliptic equation (for example, see [18]) and according to the computations in [28], we know that, for every $0 \leq t \leq T$, $l > 1$,

$$\mathbb{E} \left(\sup_{t \leq s \leq T} (|Y_s^t(x)|^l + |Z_s^t(x)|^l) \right) < \infty.$$

Then taking the expectation in (2.5) and noticing that $Y_t^t(x)$ is non-random (see [28]), we have,

$$\boxed{\text{e12a}} \quad (3.19) \quad g(T-t, x) = Y_t^t(x) = \mathbb{E}(u_0(X_T^t(x))) + \int_t^T \mathbb{E}(F_v(T-s, X_s^t(x))) ds.$$

Since $F_v(t) \in B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d) \cap C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$ for every t , we can change the order of expectation and differential, applying Hölder's inequality, to obtain

$$\|\mathbb{E}(F_v(T-s, X_s^t(\cdot)))\|_{B_{p,p}^{2+\alpha}}^p \leq \mathbb{E} \left(\|F_v(T-s, X_s^t(\cdot))\|_{B_{p,p}^{2+\alpha}}^p \right).$$

Hence by Lemma 2.2 and 3.1, for every $0 \leq t \leq s \leq T$,

$$\begin{aligned} & \|\mathbb{E}(F_v(T-s, X_s^t(\cdot)))\|_{B_{p,p}^{2+\alpha}} \\ & \leq C e^{CKT} (1 + T^2 K^2) \|F_v(T-s)\|_{B_{p,p}^{2+\alpha}} \leq CK e^{CKT} (1 + T^2 K^2) \|v(T-s)\|_{B_{p,p}^{2+\alpha}}. \end{aligned}$$

Similarly, for every $0 \leq t \leq T$,

$$\|\mathbb{E}(u_0(X_T^t(\cdot)))\|_{B_{p,p}^{2+\alpha}} \leq C e^{CKT} (1 + T^2 K^2) \|u_0\|_{B_{p,p}^{2+\alpha}}.$$

Putting the above estimate into (3.19), conclusion (3.18) follows. \square

Consider vector fields $v_m \in \mathcal{S}(p, p, p', T)$, where $d < p < \infty$, $1 < p' < \frac{d}{2}$, $0 < T < 1$, $m = 1, 2$. From now on in this section, let (X_m, Y_m, Z_m) , where $m = 1, 2$, be the solutions of (2.5) with coefficients v_m and initial condition $u_{0,m} := v_m(0)$. We will present some estimates on the difference between X_1 and X_2 .

$\boxed{19}$ **Lemma 3.4.** *For every $f_1, f_2 \in W^{1,p}(\mathbb{R}^d)$ (recall that $p > d$), $0 \leq t \leq s \leq T$, we have,*

$$\boxed{\text{e31}} \quad (3.20) \quad \begin{aligned} & \int_{\mathbb{R}^d} |f_1(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))|^p dx \\ & \leq C_1 \|f_1 - f_2\|_{L^p}^p + C_1 T^p e^{C_1 KT} \|\nabla f_2\|_{L^p}^p \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{L^\infty}^p \text{ a.s.,} \end{aligned}$$

where $K := \sup_{t \in [0, T], m=1,2} \|\nabla v_m(t)\|_{L^\infty}$, and C_1 is a constant independent of ν , K , T , p' , f_m and v_m .

Proof. We first assume $f_1, f_2 \in C^2(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$. Since

$$|f_1(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))| \leq |f_1(X_{s,1}^t(x)) - f_2(X_{s,1}^t(x))| + |f_2(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))|,$$

by (3.3), for every $0 \leq t \leq s \leq T$, we have

$$\boxed{\text{e16}} \quad (3.21) \quad \int_{\mathbb{R}^d} |f_1(X_{s,1}^t(x)) - f_2(X_{s,1}^t(x))|^p dx \leq \|f_1 - f_2\|_{L^p}^p, \quad a.s..$$

For every $r \in [1, 2]$, we define $X_s^{t,r}(x)$ to be the solution of following SDE,

$$\boxed{\text{e33}} \quad (3.22) \quad \begin{cases} dX_s^{t,r}(x) = \sqrt{2\nu} dB_s - ((2-r)v_1(T-s, X_s^{t,r}(x)) + (r-1)v_2(T-s, X_s^{t,r}(x)))ds, \\ X_t^{t,r}(x) = x, \quad 0 \leq t \leq s \leq T. \end{cases}$$

In particular $X_s^{t,r}(x) = X_{s,1}^t(x)$ when $r = 1$ and $X_s^{t,r}(x) = X_{s,2}^t(x)$ when $r = 2$. Since $\nabla \cdot ((2-r)v_1 + (r-1)v_2) = 0$, we have, for any $h \in L^1(\mathbb{R}^d)$,

$$\boxed{\text{e31a}} \quad (3.23) \quad \int_{\mathbb{R}^d} h(X_s^{t,r}(x)) dx = \int_{\mathbb{R}^d} h(x) dx, \quad \forall r \in [1, 2], \quad a.s..$$

Since v_m is regular enough, the methods of [22] allow us to check that there is a version of $X_s^{t,r}(x)$ which is differentiable with r and that $V_s^{t,r}(x) := \frac{d}{dr} X_s^{t,r}(x)$ satisfies the following SDE,

$$\boxed{\text{e33a}} \quad (3.24) \quad \begin{cases} dV_s^{t,r}(x) = -((2-r)\nabla v_1(T-s, X_s^{t,r}(x)) + (r-1)\nabla v_2(T-s, X_s^{t,r}(x)))V_s^{t,r}(x)ds \\ + (v_1(T-s, X_s^{t,r}(x)) - v_2(T-s, X_s^{t,r}(x)))ds, \\ V_t^{t,r}(x) = 0, \quad 0 \leq t \leq s \leq T. \end{cases}$$

By Grownwall lemma for every $0 \leq t \leq s \leq T$, $r \in [1, 2]$ and $x \in \mathbb{R}^d$,

$$\boxed{\text{e32}} \quad (3.25) \quad |V_s^{t,r}(x)| \leq CT e^{CKT} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{L^\infty}, \quad a.s..$$

Since

$$|f_2(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))|^p = \left| \int_1^2 \frac{d}{dr} (f_2(X_s^{t,r}(x))) dr \right|^p \leq \int_1^2 |\nabla f_2(X_s^{t,r}(x))|^p |V_s^{t,r}(x)|^p dr,$$

then, by (3.23) and (3.25), we have,

$$\begin{aligned} & \int_{\mathbb{R}^d} |f_2(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))|^p dx \\ \boxed{\text{e16a}} \quad (3.26) \quad & \leq CT^p e^{CKT} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{L^\infty}^p \int_1^2 \int_{\mathbb{R}^d} |\nabla f_2(X_s^{t,r}(x))|^p dx dr \\ & \leq CT^p e^{CKT} \int_{\mathbb{R}^d} |\nabla f_2(x)|^p dx \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{L^\infty}^p, \quad a.s.. \end{aligned}$$

Combining (3.21) and (3.26) together, we have (3.20).

For general $f_1, f_2 \in W^{1,p}(\mathbb{R}^d)$, there exist sequences $\{f_{1,n}\}_{n=1}^\infty, \{f_{2,n}\}_{n=1}^\infty \subseteq C^2(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$, such that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} |f_{i,n}(x) - f_i(x)| \leq \lim_{n \rightarrow \infty} \|f_{i,n} - f_i\|_{W^{1,p}} = 0, \quad \sup_n \|f_{i,n}\|_{W^{1,p}} \leq \|f_i\|_{W^{1,p}}, \quad i = 1, 2,$$

then according to Fatou lemma,

$$\begin{aligned} \int_{\mathbb{R}^d} |f_1(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))|^p dx &= \int_{\mathbb{R}^d} \lim_{n \rightarrow \infty} |f_{1,n}(X_{s,1}^t(x)) - f_{2,n}(X_{s,2}^t(x))|^p dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} |f_{1,n}(X_{s,1}^t(x)) - f_{2,n}(X_{s,2}^t(x))|^p dx \\ &\leq C \|f_1 - f_2\|_{L^p}^p + CT^p e^{C_1 KT} \|\nabla f_2\|_{L^p}^p \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{L^\infty}^p \text{ a.s..} \end{aligned}$$

□

110 **Lemma 3.5.** Suppose that $f_1, f_2 \in B_{p,p}^{1+\alpha}(\mathbb{R}^d)$ where $d < p < \infty$, $0 < \alpha < 1$. We have, for every $0 \leq t \leq s \leq T$,

e32a (3.27)

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| (f_1(X_{s,1}^t(x+y)) - f_2(X_{s,2}^t(x+y))) - (f_1(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))) \right|^p}{|y|^{3+p\alpha}} dx dy \\ &\leq C_1 e^{C_1 KT} [f_1 - f_2]_{B_{p,p}^\alpha}^p + C_1 T^p (1 + K^p) e^{C_1 KT} \|f_2\|_{B_{p,p}^{1+\alpha}}^p \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p \text{ a.s.,} \end{aligned}$$

where $K := \sup_{t \in [0, T]} \|v_m(t)\|_{B_{p,p}^{2+\alpha}}$, and C_1 is a positive constant independent of ν , K , T , p' , f_m , and v_m .

Proof. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $B_{p,p}^{1+\alpha}(\mathbb{R}^d)$ (see [32, Theorem 2.3.2(a)]), by the same approximation argument in the proof of Lemma 3.4, it suffices to show (3.27) for every $f_1, f_2 \in C^2(\mathbb{R}^d) \cap B_{p,p}^{1+\alpha}(\mathbb{R}^d)$.

We have,

$$\begin{aligned} &\left| (f_1(X_{s,1}^t(x+y)) - f_2(X_{s,2}^t(x+y))) - (f_1(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))) \right| \\ &\leq \left| (f_1(X_{s,1}^t(x+y)) - f_1(X_{s,1}^t(x))) - (f_2(X_{s,1}^t(x+y)) - f_2(X_{s,1}^t(x))) \right| \\ &\quad + \left| (f_2(X_{s,1}^t(x+y)) - f_2(X_{s,1}^t(x))) - (f_2(X_{s,2}^t(x+y)) - f_2(X_{s,2}^t(x))) \right| \\ &:= I_{s,1}^t(x, y) + I_{s,2}^t(x, y). \end{aligned}$$

Note that

$$I_{s,1}^t(x, y) = (f_1 - f_2)(X_{s,1}^t(x+y)) - (f_1 - f_2)(X_{s,1}^t(x)),$$

by property (3.15) in the proof of Lemma 3.1, we have,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_{s,1}^t(x, y)|^p}{|y|^{3+p\alpha}} dx dy \leq C e^{CKT} \|f_1 - f_2\|_{B_{p,p}^\alpha}, a.s..$$

For $r \in [0, 1]$, let $X_s^{t,r}(x)$, $V_s^{t,r}(x)$ be the solutions of SDEs (3.22) and (3.24) respectively. Hence

$$\begin{aligned} |I_{s,2}^t(x, y)|^p &= \left| \int_1^2 \frac{d}{dr} (f_2(X_s^{t,r}(x+y)) - f_2(X_s^{t,r}(x))) dr \right|^p \\ \text{e34} \quad (3.28) \quad &\leq C \int_1^2 |\nabla f_2(X_s^{t,r}(x+y))|^p |V_s^{t,r}(x+y) - V_s^{t,r}(x)|^p dr \\ &+ C \int_1^2 |\nabla f_2(X_s^{t,r}(x+y)) - \nabla f_2(X_s^{t,r}(x))|^p |V_s^{t,r}(x)|^p dr. \end{aligned}$$

Let $\Gamma_s^{t,r}(x, y) := V_s^{t,r}(x+y) - V_s^{t,r}(x)$. Applying Itô's formula in (3.24), we derive that

$$\begin{aligned} |\Gamma_s^{t,r}(x, y)| &\leq K \int_s^T |\Gamma_u^{t,r}(x, y)| du \\ \text{e14a} \quad (3.29) \quad &+ \int_s^T |\nabla v_r(T-u, X_u^{t,r}(x+y)) - \nabla v_r(T-u, X_u^{t,r}(x))| |V_u^{t,r}(x)| du \\ &+ \int_s^T |(v_2 - v_1)(T-u, X_u^{t,r}(x+y)) - (v_2 - v_1)(T-u, X_u^{t,r}(x))| du \end{aligned}$$

where $v_r(s, x) := (2-r)\nabla v_1(s, x) + (r-1)\nabla v_2(s, x)$.

In the same way we proved (3.5) and (3.10), for every $r \in [1, 2]$, $0 \leq t \leq s \leq T$, $x, y \in \mathbb{R}^d$,

$$\text{e34a} \quad (3.30) \quad |X_s^{t,r}(x+y) - X_s^{t,r}(x)| \leq C e^{CKT} |y|, \quad a.s..$$

Hence according to (2.9) and (3.30),

$$\begin{aligned} &|\nabla v_r(T-u, X_u^{t,r}(x+y)) - \nabla v_r(T-u, X_u^{t,r}(x))| \\ &\leq C e^{CKT} \|v_r(T-u)\|_{B_{p,p}^{2+\alpha}} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) \\ &\leq CK e^{CKT} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}), \quad a.s.. \end{aligned}$$

As in (2.9), using the embedding theorem [32, Theorem 2.8.1] and (3.30), we can show that for every $r \in [1, 2]$, $0 \leq t \leq u \leq T$,

$$\begin{aligned} &|(v_2 - v_1)(T-u, X_u^{t,r}(x+y)) - (v_2 - v_1)(T-u, X_u^{t,r}(x))| \\ &\leq C e^{CKT} \|v_1(T-u) - v_2(T-u)\|_{B_{p,p}^{1+\alpha}} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}), \quad a.s.. \end{aligned}$$

Combining the above estimate and (3.25) together into (3.29), we have,

$$\boxed{\text{e13a}} \quad (3.31) \quad |\Gamma_s^{t,r}(x, y)| \leq CT(1 + KT)e^{CKT} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}), \text{ a.s..}$$

Analogously to the proof of (3.15) in Lemma 3.1, we can show that for every $r \in [1, 2]$,

$$\boxed{\text{e14}} \quad (3.32) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla f_2(X_s^{t,r}(x+y)) - \nabla f_2(X_s^{t,r}(x))|^p}{|y|^{d+\alpha p}} dx dy \leq Ce^{CKT} \|f_2\|_{B_{p,p}^{1+\alpha}}^p, \text{ a.s..}$$

Combining (3.31) and (3.32) into (3.28) and using properties (3.23), (3.25), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_{s,2}^t(x, y)|^p}{|y|^{3+p\alpha}} dx dy \\ & \leq CT^p(1 + K^p T^p) e^{CKT} \|f_2\|_{B_{p,p}^{1+\alpha}}^p \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p. \end{aligned}$$

Since we assume that $0 < T < 1$, by putting the estimates for $I_{s,1}^t$ and $I_{s,2}^t$ together, we can prove (3.27). \square

Based on Lemma 3.4 and 3.5, we can prove the following difference estimate.

$\boxed{\text{111}}$ **Lemma 3.6.** *Suppose that $f_1, f_2 \in B_{p,p}^{2+\alpha}(\mathbb{R}^d)$ for some $d < p < \infty$, $0 < \alpha < 1$. Then for every $0 \leq t \leq s \leq T$,*

$$\boxed{\text{e15}} \quad (3.33) \quad \begin{aligned} & \|f_1 \circ X_{s,1}^t(\cdot) - f_2 \circ X_{s,2}^t(\cdot)\|_{B_{p,p}^{1+\alpha}} \leq C_1 e^{C_1 KT} (1 + TK) \|f_1 - f_2\|_{B_{p,p}^{1+\alpha}} \\ & + C_1 T e^{C_1 KT} (1 + T^2 K^2) \|f_2\|_{B_{p,p}^{2+\alpha}} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}} \text{ a.s.,} \end{aligned}$$

where $K := \sup_{t \in [0, T]} \|v_m(t)\|_{B_{p,p}^{2+\alpha}}$, and C_1 is a positive constant independent of ν , K , T , p' , f_m and v_m .

Proof. Step 1: By Lemma 3.4 and applying Sobolev embedding theorem, for every $0 \leq t \leq s \leq T$,

$$\boxed{\text{111.1}} \quad (3.34) \quad \begin{aligned} & \int_{\mathbb{R}^d} |f_1(X_{s,1}^t(x)) - f_2(X_{s,2}^t(x))|^p dx \\ & \leq C \|f_1 - f_2\|_{L^p}^p + CT^p e^{CKT} \|f_2\|_{W^{1,p}}^p \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}^p, \text{ a.s..} \end{aligned}$$

Note that for $m = 1, 2$, $\nabla(f_m \circ X_{s,m}^t)(x) = \nabla f_m(X_{s,m}^t(x)) \nabla X_{s,m}^t(x)$, we have,

$$\boxed{\text{e15a}} \quad (3.35) \quad \begin{aligned} & |\nabla(f_1 \circ X_{s,1}^t)(x) - \nabla(f_2 \circ X_{s,2}^t)(x)| \\ & \leq Ce^{CKT} |\nabla f_1(X_{s,1}^t(x)) - \nabla f_2(X_{s,2}^t(x))| + C \|\nabla f_2\|_{L^\infty} |\nabla X_{s,1}^t(x) - \nabla X_{s,2}^t(x)|, \end{aligned}$$

where we use the estimate (3.5). Applying (3.34), we have,

$$\begin{aligned} & \int_{\mathbb{R}^d} |\nabla f_1(X_{s,1}^t(x)) - \nabla f_2(X_{s,2}^t(x))|^p dx \\ & \leq C \|f_1 - f_2\|_{W^{1,p}}^p + CT^p e^{CKT} \|f_2\|_{W^{2,p}}^p \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}^p, \text{ a.s..} \end{aligned}$$

Let $\Gamma_s^t(x) := \nabla X_{s,1}^t(x) - \nabla X_{s,2}^t(x)$. Applying Itô's formula in the first equation in (3.2) and using the estimate (3.5), we get that

$$|\Gamma_s^t(x)| \leq CK \int_t^s |\Gamma_r^t(x)| dr + Ce^{CKT} \int_t^s |\nabla v_1(T-r, X_{r,1}^t(x)) - \nabla v_2(T-r, X_{r,2}^t(x))| dr,$$

hence, by applying (3.34) to $f_1 = \nabla v_1$, $f_2 = \nabla v_2$ and using Grownwall lemma, together with Hölder's inequality, we may obtain

$$\boxed{\text{e17a}} \quad (3.36) \quad \int_{\mathbb{R}^d} |\Gamma_s^t(x)|^p dx \leq CT^p e^{CKT} (1 + T^p K^p) \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}^p \text{ a.s..}$$

Putting the above estimates into (3.35) and noticing that $0 < T < 1$, we have,

$$\boxed{111.2} \quad (3.37) \quad \begin{aligned} & \int_{\mathbb{R}^d} |\nabla(f_1 \circ X_{s,1}^t)(x) - \nabla(f_2 \circ X_{s,2}^t)(x)|^p dx \\ & \leq Ce^{CKT} \|f_1 - f_2\|_{W^{1,p}}^p + CT^p e^{CKT} (1 + T^p K^p) \|f_2\|_{W^{2,p}}^p \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}^p, \text{ a.s..} \end{aligned}$$

Step 2: By (2.13), for every $0 \leq t \leq s \leq T$, $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \boxed{\text{e17}} \quad (3.38) \quad & \left| (\nabla(f_1 \circ X_{s,1}^t)(x+y) - \nabla(f_2 \circ X_{s,2}^t)(x+y)) - (\nabla(f_1 \circ X_{s,1}^t)(x) - \nabla(f_2 \circ X_{s,2}^t)(x)) \right| \\ & \leq Ce^{CKT} \left| (\nabla f_1(X_{s,1}^t(x+y)) - \nabla f_2(X_{s,2}^t(x+y))) - (\nabla f_1(X_{s,1}^t(x)) - \nabla f_2(X_{s,2}^t(x))) \right| \\ & + C \|\nabla f_1\|_{L^\infty} |(\nabla X_{s,1}^t(x+y) - \nabla X_{s,2}^t(x+y)) - (\nabla X_{s,1}^t(x) - \nabla X_{s,2}^t(x))| \\ & + C |\nabla X_{s,1}^t(x+y) - \nabla X_{s,2}^t(x+y)| |\nabla f_2(X_{s,2}^t(x+y)) - \nabla f_2(X_{s,2}^t(x))| \\ & + C |\nabla X_{s,2}^t(x+y) - \nabla X_{s,2}^t(x)| |\nabla f_1(X_{s,1}^t(x)) - \nabla f_2(X_{s,2}^t(x))| \\ & := \sum_{i=1}^4 I_{s,i}^t(x, y), \end{aligned}$$

where we have used estimate (3.5). According to Lemma 3.5,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_{s,1}^t(x, y)|^p}{|y|^{d+\alpha p}} dy dx \\ & \leq Ce^{CKT} \|f_1 - f_2\|_{B_{p,p}^{1+\alpha}} + CT^p e^{CKT} (1 + K^p) \|f_2\|_{B_{p,p}^{2+\alpha}}^p \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p \text{ a.s..} \end{aligned}$$

From (2.9) and (3.10),

$$\begin{aligned}
& |\nabla f_2(X_{s,2}^t(x+y)) - \nabla f_2(X_{s,2}^t(x))| \\
& \leq C \|f_2\|_{B_{p,p}^{2+\alpha}} \left(|X_{s,2}^t(x+y) - X_{s,2}^t(x)|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}} \right) \\
& \leq C e^{CKT} \|f_2\|_{B_{p,p}^{2+\alpha}} (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}), \quad a.s.
\end{aligned}$$

Combining this with (3.36), we deduce that

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_{s,3}^t(x,y)|^p}{|y|^{d+\alpha p}} dy dx \\
& \leq CT^p e^{CKT} (1 + T^p K^p) \|f_2\|_{B_{p,p}^{2+\alpha}}^p \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}^p \quad a.s..
\end{aligned}$$

According to estimate (3.13) and Lemma 3.4,

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_{s,4}^t(x,y)|^p}{|y|^{d+\alpha p}} dy dx \\
& \leq CT^p K^p e^{CKT} \left(\|f_1 - f_2\|_{W^{1,p}}^p + T^p \|f_2\|_{W^{2,p}}^p \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}^p \right) \quad a.s..
\end{aligned}$$

Let $\Psi_s^t(x,y) := (\nabla X_{s,1}^t(x+y) - \nabla X_{s,2}^t(x+y)) - (\nabla X_{s,1}^t(x) - \nabla X_{s,2}^t(x))$. Applying Itô's formula in (3.2), and estimate (3.5), we have

$$\begin{aligned}
& |\Psi_s^t(x,y)| \leq CK \int_t^s |\Psi_r^t(x,y)| dr \\
& + C e^{CKT} \int_t^s |(\nabla v_1(T-r, X_{r,1}^t(x+y)) - \nabla v_2(T-r, X_{r,2}^t(x+y))) \\
& - (\nabla v_1(T-r, X_{r,1}^t(x)) - \nabla v_2(T-r, X_{r,2}^t(x)))| dr \\
& + C \int_t^s |\nabla X_{r,1}^t(x+y) - \nabla X_{r,2}^t(x+y)| |\nabla v_2(T-r, X_{r,2}^t(x+y)) - \nabla v_2(T-r, X_{r,2}^t(x))| dr \\
& + C \int_t^s |\nabla X_{r,2}^t(x+y) - \nabla X_{r,2}^t(x)| |\nabla v_1(T-r, X_{r,1}^t(x)) - \nabla v_2(T-r, X_{r,2}^t(x))| dr.
\end{aligned}$$

As in the previous argument first applying Grownwall lemma, then using Hölder inequality and estimating the drift terms in the same way as for $I_{s,i}^t(x,y)$, $i = 1, 3, 4$, we may obtain

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Psi_s^t(x,y)|^p}{|y|^{d+\alpha p}} dy dx \\
& \leq CT^p e^{CKT} (1 + T^{2p} K^{2p}) \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p \quad a.s.,
\end{aligned}$$

where we have used the assumption that $0 < T < 1$. Plugging the estimates obtained above together into (3.38), we obtain for every $0 \leq t \leq s \leq T$ that

$$\boxed{111.3} \quad (3.39) \quad \begin{aligned} [f_1 \circ X_{s,1}^t - f_2 \circ X_{s,2}^t]_{B_{p,p}^{1+\alpha}}^p &\leq C e^{CKT} (1 + T^p K^p) \|f_1 - f_2\|_{B_{p,p}^{1+\alpha}}^p \\ &+ CT^p e^{CKT} (1 + T^{2p} K^{2p}) \|f_2\|_{B_{p,p}^{2+\alpha}}^p \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p \text{ a.s.} \end{aligned}$$

Putting (3.34), (3.37) and (3.39) together to conclude the proof. \square

We are now in a position to prove the following difference estimate which will be used to prove the local existence.

$\boxed{112}$ **Lemma 3.7.** *Let $g_m(t, x) := Y_{t,m}^t(x)$, $m = 1, 2$, where $d < p < \infty$, $0 < \alpha < 1$, $0 \leq t \leq T$. Then*

$$\boxed{e35} \quad (3.40) \quad \begin{aligned} \sup_{t \in [0,T]} \|g_1(t) - g_2(t)\|_{B_{p,p}^{1+\alpha}} &\leq C_1 e^{C_1 KT} (1 + TK) \|u_{0,1} - u_{0,2}\|_{B_{p,p}^{1+\alpha}} \\ &+ CT K (1 + T^3 K^3) e^{CKT} \sup_{t \in [0,T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}, \end{aligned}$$

where $K := \sup_{t \in [0,T], m=1,2} \|v_m(t)\|_{B_{p,p}^{2+\alpha}}$, and C_1 is a positive constant independent of K, ν, T, p' and v_m .

Proof. As (3.19), for $m = 1, 2$, $0 \leq t \leq T$,

$$\boxed{e18a} \quad (3.41) \quad g_m(t) = Y_{t,m}^t(x) = \mathbb{E}(u_{0,m}(X_{T,m}^t(x))) + \int_t^T \mathbb{E}(F_{v_m}(T-s, X_{s,m}^t(x))) ds.$$

According to the regularity theorem of elliptic equation, $F_v(t) \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$ for every t , so that we can change the order of expectation and differential, together with Hölder's inequality, to obtain

$$\boxed{e18} \quad (3.42) \quad \begin{aligned} &\|\mathbb{E}(F_{v_1}(T-s, X_{s,1}^t(\cdot))) - \mathbb{E}(F_{v_2}(T-s, X_{s,2}^t(\cdot)))\|_{B_{p,p}^{1+\alpha}}^p \\ &\leq \mathbb{E}\left(\|F_{v_1}(T-s, X_{s,1}^t(\cdot)) - F_{v_2}(T-s, X_{s,2}^t(\cdot))\|_{B_{p,p}^{1+\alpha}}^p\right). \end{aligned}$$

By Lemma 3.6, for every $0 \leq t \leq s \leq T$,

$$\begin{aligned} &\|F_{v_1}(T-s, X_{s,1}^t(\cdot)) - F_{v_2}(T-s, X_{s,2}^t(\cdot))\|_{B_{p,p}^{1+\alpha}}^p \\ &\leq C e^{CKT} (1 + T^p K^p) \|F_{v_1}(T-s) - F_{v_2}(T-s)\|_{B_{p,p}^{1+\alpha}}^p \\ &+ CT^p e^{CKT} (1 + T^{2p} K^{2p}) \sup_{t \in [0,T]} \|F_{v_2}(T-s)\|_{B_{p,p}^{2+\alpha}}^p \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p \text{ a.s.,} \end{aligned}$$

thus, according to Lemmas 2.2, 2.4,

$$\begin{aligned} & \|F_{v_1}(T-s, X_{s,1}^t(\cdot)) - F_{v_2}(T-s, X_{s,2}^t(\cdot))\|_{B_{p,p}^{1+\alpha}}^p \\ & \leq CK^p e^{CKT} (1 + T^{3p} K^{3p}) \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}^p \text{ a.s.} \end{aligned}$$

Putting this into (3.42),

$$\begin{aligned} & \|\mathbb{E}(F_{v_1}(T-s, X_{s,1}^t(\cdot))) - \mathbb{E}(F_{v_2}(T-s, X_{s,2}^t(\cdot)))\|_{B_{p,p}^{1+\alpha}} \\ & \leq CK e^{CKT} (1 + T^3 K^3) \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}. \end{aligned}$$

Similarly, we can show that,

$$\begin{aligned} & \|\mathbb{E}(u_{0,1}(X_{T,1}^t(\cdot))) - \mathbb{E}(u_{0,2}(X_{T,2}^t(\cdot)))\|_{B_{p,p}^{1+\alpha}} \\ & \leq C e^{CKT} (1 + TK) \|u_{0,1} - u_{0,2}\|_{B_{p,p}^{1+\alpha}} + CTK(1 + T^2 K^2) e^{CKT} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{1+\alpha}}. \end{aligned}$$

Putting the above estimate into (3.41), we have (3.40). \square

r1 **Remark 3.8.** Note that in the estimate (3.40) for $\|g_1(t) - g_2(t)\|_{B_{p,p}^{1+\alpha}}$, the difference term $\|v_1 - v_2\|_{B_{p,p}^{1+\alpha}}$ is considered with the $B_{p,p}^{1+\alpha}$ norm, and the uniformly control term is with the $B_{p,p}^{2+\alpha}$ norm (see the definition of K), which is one order higher. But in the estimate (3.18) for $\|g(t)\|_{B_{p,p}^{2+\alpha}}$, only $B_{p,p}^{2+\alpha}$ norm is involved, which is of the same order as the one of $g(t)$.

Since for $p > 1$ and $r > 1 + \frac{d}{p}$, $\|\nabla v(t)\|_{L^\infty} \leq C\|v(t)\|_{B_{p,q}^r}$, repeating the procedure used above, we can also obtain the following estimate with lower and higher order Besov norm.

For every $p > 1$ and $r > 1 + \frac{d}{p}$,

$$\begin{aligned} & \sup_{t \in [0, T]} \|g_1(t) - g_2(t)\|_{B_{p,p}^{\max(r-1, 1)}} \leq C_1 e^{C_1 KT} (1 + TK^{[r]-1}) \|u_{0,1} - u_{0,2}\|_{B_{p,p}^{\max(r-1, 1)}} \\ & + CTK(1 + T^{[r]+1} K^{[r]+1}) e^{CKT} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{B_{p,p}^{\max(r-1, 1)}} \end{aligned} \quad \text{e19aa} \quad (3.43)$$

and for $m = 1, 2$,

$$\text{e19} \quad (3.44) \quad \sup_{t \in [0, T]} \|g_m(t)\|_{B_{p,p}^r} \leq C_1 (1 + T^{[r]} K^{[r]}) e^{C_1 KT} \|u_0\|_{B_{p,p}^r} + C_1 TK^2 (1 + T^{[r]+1} K^{[r]+1}),$$

where $r := [r] + [r]^+$ with $[r]$ to be an integer and $0 \leq [r]^+ < 1$, $K := \sup_{t \in [0, T], m=1,2} \|v_m(t)\|_{B_{p,p}^r}$, and C_1 is a positive constant independent of K , ν , T , p' , v_m . In particular, if r is an integer, we use the notation $\|\cdot\|_{B_{p,p}^r}$ to denote the Sobolev norm $\|\cdot\|_{W^{r,p}}$.

Note that estimate (3.18) does not yield the regularity with respect to the time variable of g , however, according to Remark 3.8, we can show that $g \in C([0, T]; B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$.

c1 **Corollary 3.9.** *Let $v, g(t)$ be as in Lemma 3.3. Then for every $r > 1 + \frac{d}{p}$, we have $g \in C([0, T]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$, and the estimate (3.44) holds for g .*

Proof. Since $v \in \mathcal{S}(p, p, p', T)$, by the elliptic regularity $F_v(t) \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Hence by [28, Theorem 3.2], $g(t, x) := Y_{T-t}^{T-t}(x) \in C^1([0, T]; C_b^2(\mathbb{R}^d; \mathbb{R}^d))$ satisfies the following parabolic PDE,

$$\frac{\partial g}{\partial t} + v \cdot \nabla g = \nu \Delta g + F_v, \quad g(0) = u_0.$$

Hence for every $0 \leq t \leq s \leq T, x \in \mathbb{R}^d$,

$$g(s, x) - g(t, x) = \int_t^s \left(\nu \Delta g(r, x) + F_v(r, x) - v(r, x) \cdot \nabla g(r, x) \right) dr,$$

so that, according to (3.44) and Lemma 2.2,

$$\|g(s) - g(t)\|_{B_{p,p}^r} \leq C(s-t) \left(1 + \sup_{t \in [0, T]} (\|g(t)\|_{B_{p,p}^{r+2}}^2 + \|v(t)\|_{B_{p,p}^r}^2) \right),$$

which implies that $g \in C([0, T]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$. □

As stated in Section 2, for every $v \in \mathcal{S}(p, p, p', T)$, we can define $\mathcal{J}_\nu(v)(t) := \mathbf{P}(Y_{T-t}^{T-t}(\cdot))$ for every $t \in [0, T]$, where Y is the solution of (2.5) with coefficients v and initial condition $u_0 = v(0)$, \mathbf{P} is the Leray-Hodge projection to divergence free vector fields. By Lemmas 3.3 and 3.7, the extension property of the map \mathcal{J}_ν hold.

p1 **Proposition 3.10.** *For every $p > 1, r > 1 + \frac{d}{p}, T > 0, u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ satisfying that $\nabla \cdot u_0 = 0$, \mathcal{J}_ν can be extended to be a map $\mathcal{J}_\nu : \mathcal{B}(u_0, T, p, r) \rightarrow \mathcal{B}(u_0, T, p, r)$, where*

e24aa (3.45)
$$\mathcal{B}(u_0, T, p, r) := \left\{ v \in C([0, T]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)); \quad v(0, x) = u_0(x), \right. \\ \left. \nabla \cdot v(t) = 0, \quad \forall t \in [0, T] \right\},$$

and for $v_1, v_2 \in \mathcal{B}(u_0, T, p, r)$, the estimate (3.43), (3.44) holds with g_1, g_2 replaced by $\mathcal{J}_\nu(v_1), \mathcal{J}_\nu(v_2)$.

Proof. Step 1: Since $C_c^\infty(\mathbb{R}^d)$ is dense in $B_{p,p}^r(\mathbb{R}^d)$ (see [32, Theorem 2.3.2(a)]), for every $v \in \mathcal{B}(u_0, T, p, r)$, we can find a sequence $\{\tilde{v}_n\}_{n=1}^\infty$, such that for every n , $\tilde{v}_n \in C([0, T]; C_c^\infty(\mathbb{R}^d; \mathbb{R}^d))$ (however we can not assume that $\nabla \cdot \tilde{v}_n(t) = 0$), and

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\tilde{v}_n(t) - v(t)\|_{B_{p,p}^r} = 0.$$

Let $v_n(t) := \mathbf{P}\tilde{v}_n(t)$: we have $v_n \in \mathcal{S}(p, p', T)$ for every $1 < p' < \frac{d}{2}$. Also note that \mathbf{P} is a singular integral operator, which is bounded in $B_{p,p}^r(\mathbb{R}^d)$ (see [29] or the proof of Lemma 2.2). Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|v_n(t) - v(t)\|_{B_{p,p}^r} &= \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\mathbf{P}\tilde{v}_n(t) - \mathbf{P}v(t)\|_{B_{p,p}^r} \\ &\leq C \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\tilde{v}_n(t) - v(t)\|_{B_{p,p}^r} = 0, \end{aligned}$$

in particular, for $u_{0,n} := v_n(0)$,

$$\boxed{\text{e20}} \quad (3.46) \quad \lim_{n \rightarrow \infty} \|u_{0,n} - u_0\|_{B_{p,p}^r} = 0.$$

Since \mathbf{P} is a singular integral operator, by (3.44) and (3.43), $\{\mathcal{J}_\nu(v_n)\}_{n=1}^\infty$ is a Cauchy sequence in $C([0, T]; B_{p,p}^{\max(r-1, 1)}(\mathbb{R}^d; \mathbb{R}^d))$ and there is a $\hat{v} \in C([0, T]; B_{p,p}^{\max(r-1, 1)}(\mathbb{R}^d; \mathbb{R}^d))$ such that,

$$\boxed{\text{e20a}} \quad (3.47) \quad \sup_n \sup_{t \in [0, T]} \|\mathcal{J}_\nu(v_n)(t)\|_{B_{p,p}^r} < \infty,$$

$$\boxed{\text{e21}} \quad (3.48) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\mathcal{J}_\nu(v_n)(t) - \hat{v}(t)\|_{B_{p,p}^{\max(r-1, 1)}} = 0,$$

from (3.48) we know that $\nabla \cdot \hat{v}(t) = 0$ for every t . Since $\mathcal{J}_\nu(v_n)(0) = u_{0,n}$ by definition, according to (3.48) and (3.46), we have $\hat{v}(0) = u_0$. Also note that due to (3.43), the limit \hat{v} we have obtained above is independent of the choice of approximation sequence $\{\tilde{v}_n\}$ to v , so $\mathcal{J}_\nu(v) := \hat{v}$ is well defined. And by (3.47), (3.48), we obtain immediately that for every $v_1, v_2 \in \mathcal{B}(u_0, T, p, r)$, (3.43) holds with g_1, g_2 replaced by $\mathcal{J}_\nu(v_1), \mathcal{J}_\nu(v_2)$. In order to prove that $\hat{v} \in \mathcal{B}(u_0, T, p, r)$, it only remains to show that $\hat{v} \in C([0, T]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$.

Step 2: For simplicity, we only consider the case $p > d$ and $r = 2 + \alpha$ for some $0 < \alpha < 1$, the other case can be shown similarly. Based on (3.47), (3.48), by the interpolation inequality [32, Theorem 2.4.1(a)], we have,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|f_n(t) - \hat{v}(t)\|_{W^{2,p}} = 0,$$

where $f_n(t) := \mathcal{J}_\nu(v_n)(t)$, which implies that $\hat{v} \in C([0, T]; W^{2,p}(\mathbb{R}^d; \mathbb{R}^d))$. For every fixed $t \in [0, T]$, taking a subsequence if necessary, we have

$$\lim_{n \rightarrow \infty} \nabla^2 f_n(t) = \nabla^2 \hat{v}(t), \text{ a.e.,}$$

where *a.e.* means the almost everywhere with respect to the Lebesgue measure. Hence by Fatou lemma,

$$\begin{aligned}
[\hat{v}(t)]_{B_{p,p}^{2+\alpha}}^p &= \int_{\mathbb{R}^d} \frac{\|\nabla^2 \hat{v}(t, \cdot + y) - \nabla^2 \hat{v}(t, \cdot)\|_{L^p}^p}{|y|^{d+\alpha p}} dy \\
&\leq \int_{\mathbb{R}^d} \frac{\liminf_{n \rightarrow \infty} \|\nabla^2 f_n(t, \cdot + y) - \nabla^2 f_n(t, \cdot)\|_{L^p}^p}{|y|^{d+\alpha p}} dy \\
&\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} \frac{\|\nabla^2 f_n(t, \cdot + y) - \nabla^2 f_n(t, \cdot)\|_{L^p}^p}{|y|^{d+\alpha p}} dy \\
&\leq \sup_n \sup_{t \in [0, T]} \|f_n(t)\|_{B_{p,p}^{2+\alpha}}^p < \infty,
\end{aligned}$$

so we obtain $\hat{v} \in L^\infty([0, T]; B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$, and for every $v \in \mathcal{B}(u_0, T, p, r)$, (3.44) holds with g replaced by $\mathcal{J}_\nu(v)$.

Step 3: Note that $v \in C([0, T]; B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$, the backward SDE (2.5) has a unique solution. Let (X_s^t, Y_s^t, Z_s^t) , $(X_{s,n}^t, Y_{s,n}^t, Z_{s,n}^t)$ be the solution of (2.5) with the coefficients v and v_n respectively. Since $v(t) \in C_b^{1,r(p)}(\mathbb{R}^d; \mathbb{R}^d)$, we know that $X_s^t(\cdot)$ has a version which is differentiable with respect to x , and the derivative ∇X_s^t satisfies the following,

$$\boxed{\text{e28c}} \quad (3.49) \quad \nabla X_s^t(x) = \mathbf{I} + \int_t^s \nabla v(T-r, X_r^t(x)) \nabla X_r^t(x) dr.$$

Then following the same arguments of step 1 in the proof of Lemma 3.6 (especially the one for (3.36)), we derive $\nabla X_s^t \in L_{loc}^p(\mathbb{R}^d; \mathbb{R}^d)$,

$$\boxed{\text{e21aaa}} \quad (3.50) \quad \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq s \leq T} \|\nabla X_{s,n}^t - \nabla X_s^t\|_{L^p} = 0.$$

By (3.16), $\sup_n \sup_{0 \leq t \leq s \leq T} \|\nabla^2 X_{s,n}^t\|_{B_{p,p}^\alpha} < \infty$, which implies that $\sup_{n,k} \sup_{0 \leq t \leq s \leq T} \|\nabla X_{s,n}^t - \nabla X_{s,k}^t\|_{B_{p,p}^{1+\alpha}} < \infty$ (note that we only obtain $\nabla X_{s,n}^t - \nabla X_{s,k}^t \in L^p(\mathbb{R}^d; \mathbb{R}^d)$, but we may not have $\nabla X_{s,n}^t \in L^p(\mathbb{R}^d; \mathbb{R}^d)$); combining this with (3.50) and by interpolation inequality, we obtain $\nabla X_s^t \in W_{loc}^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq s \leq T} \|\nabla X_{s,n}^t - \nabla X_s^t\|_{W^{1,p}} = 0,$$

and we have the following expression,

$$\boxed{\text{e21a}} \quad (3.51) \quad \nabla^2 X_s^t(x) = \int_t^s \nabla v(T-r, X_r^t(x)) \nabla^2 X_r^t(x) dr + \int_t^s \nabla^2 v(T-r, X_r^t(x)) (\nabla X_r^t(x))^2 dr,$$

then it is easy to show for every $0 \leq t \leq s \leq T$,

$$\lim_{t' \rightarrow t} \|\nabla^2 X_s^{t'} - \nabla^2 X_s^t\|_{L^p} = 0,$$

and following the same procedure as in (3.16), we have $\sup_{0 \leq t \leq s \leq T} \|\nabla X_s^t\|_{B_{p,p}^{1+\alpha}} < \infty$.

Step 4: Now we want to show that for every $0 \leq t \leq s \leq T$,

$$\boxed{\text{e29c}} \quad (3.52) \quad \lim_{t' \rightarrow t} \mathbb{E}([\nabla^2 X_s^{t'} - \nabla^2 X_s^t]_{B_{p,p}^\alpha}) = 0.$$

We first prove for every $h \in B_{p,p}^\alpha(\mathbb{R}^d)$, $0 \leq t \leq s \leq T$,

$$\boxed{\text{e28aa}} \quad (3.53) \quad \lim_{t' \rightarrow t} [h \circ X_s^{t'} - h \circ X_s^t]_{B_{p,p}^\alpha} = 0, \text{ a.s..}$$

Without loss of generality, we assume $0 \leq t \leq t' \leq s \leq T$. From (2.5), let $\Gamma_s^{t,t'}(x) := X_s^t(x) - X_s^{t'}(x)$, $K := \sup_{t \in [0,T]} \|v(t)\|_{B_{p,p}^{2+\alpha}}$, it is easy to see that,

$$\begin{aligned} |\Gamma_s^{t,t'}(x)| &\leq |X_{t'}^t(x) - x| + \int_{t'}^s K |\Gamma_r^{t,t'}(x)| dr \\ &\leq K|t' - t| + \sqrt{2\nu}|B_{t'} - B_t| + \int_{t'}^s K |\Gamma_r^{t,t'}(x)| dr, \end{aligned}$$

so by Grownwall lemma, for every $x \in \mathbb{R}^d$,

$$\boxed{\text{e26aa}} \quad (3.54) \quad |\Gamma_s^{t,t'}(x)| \leq C e^{CKT} (K|t' - t| + \sqrt{2\nu}|B_{t'} - B_t|)$$

If $h \in C_c^\infty(\mathbb{R}^d)$, since $|X_s^t(x+y) - X_s^t(x)| \leq C e^{CKT} |y|$,

$$\begin{aligned} &\left| h(X_s^t(x+y)) - h(X_s^t(x)) - (h(X_s^{t'}(x+y)) - h(X_s^{t'}(x))) \right| \\ &\leq C e^{CKT} \|\nabla h\|_{L^\infty} |y| 1_{\{|y| \leq 1\}} + C \|h\|_{L^\infty} 1_{\{|y| > 1\}}. \end{aligned}$$

By (3.54) and the dominated convergence theorem,

$$\lim_{t' \rightarrow t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|h(X_s^t(x+y)) - h(X_s^t(x)) - (h(X_s^{t'}(x+y)) - h(X_s^{t'}(x)))|^p}{|y|^{d+\alpha p}} dx dy = 0,$$

which implies that for every $h \in C_c^\infty(\mathbb{R}^d)$,

$$\boxed{\text{e26c}} \quad (3.55) \quad \lim_{t' \rightarrow t} [h \circ X_s^{t'} - h \circ X_s^t]_{B_{p,p}^\alpha} = 0.$$

Since $C_c^\infty(\mathbb{R}^d)$ is dense in $B_{p,p}^\alpha(\mathbb{R}^d)$, there exists a sequence $\{h_n\}_{n=1}^\infty \subseteq C_c^\infty(\mathbb{R}^d)$, such that $\lim_{n \rightarrow \infty} \|h_n - h\|_{B_{p,p}^\alpha} = 0$. So by the argument in Step 3 of the proof of Lemma 3.1, we obtain,

$$\boxed{\text{e21c}} \quad (3.56) \quad \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq s \leq T} [h_n \circ X_s^t - h \circ X_s^t]_{B_{p,p}^\alpha} = 0.$$

Combining (3.55) and (3.56) together,

$$\begin{aligned} & \limsup_{t' \rightarrow t} [h \circ X_s^{t'} - h \circ X_s^t]_{B_{p,p}^\alpha} \\ & \leq \limsup_{t' \rightarrow t} C[h_n \circ X_s^{t'} - h_n \circ X_s^t]_{B_{p,p}^\alpha} + \limsup_{n \rightarrow \infty} C \sup_{0 \leq t \leq s \leq T} [h_n \circ X_s^t - h \circ X_s^t]_{B_{p,p}^\alpha} = 0, \end{aligned}$$

hence (3.53) holds.

Step 5: Let $\Lambda_s^{t,t'}(x) := \nabla X_s^{t'}(x) - \nabla X_s^t(x)$, since $|\nabla X_s^t(x)| \leq e^{CKT}$, by (3.49),

$$\begin{aligned} |\Lambda_s^{t,t'}(x)| & \leq |\nabla X_{t'}^t(x) - \mathbf{I}| + K \int_{t'}^s |\Lambda_r^{t,t'}(x)| dr \\ & + Ce^{CKT} \int_{t'}^s |\nabla v(T-r, X_r^{t'}(x)) - \nabla v(T-r, X_r^t(x))| dr \\ & \leq CKe^{CKT}|t' - t| + K \int_{t'}^s |\Lambda_r^{t,t'}(x)| dr + CKe^{CKT} \int_{t'}^s |\Gamma_r^{t,t'}(x)|^{r(p)} dr, \end{aligned}$$

where we also use (2.9), and $\Gamma_s^{t,t'}(x) := X_s^t(x) - X_s^{t'}(x)$. So by Grownwall lemma and (3.54),

e29aa

 (3.57)
$$|\Lambda_s^{t,t'}(x)| \leq CKe^{CKT}((1+T)|t' - t| + \sqrt{2\nu}T|B_{t'} - B_t|).$$

Let $\Theta_s^{t,t'}(x, y) := \nabla X_s^{t'}(x+y) - \nabla X_s^t(x) - (\nabla X_s^t(x+y) - \nabla X_s^t(x))$, then by (3.49), and applying (2.13), we get,

$$\begin{aligned} |\Theta_s^{t,t'}(x, y)| & \leq |J_{t'}^t(x, y)| + \int_{t'}^s K|\Theta_r^{t,t'}(x, y)| dr \\ & + Ce^{CKT} \int_{t'}^s |I_r^{t,t'}(x, y)| dr + CKe^{CKT} \int_{t'}^s |\Lambda_r^{t,t'}(x)| (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) dr \\ & + CK \int_{t'}^s |\Gamma_r^{t,t'}(x+y)|^{r(p)} |J_r^t(x, y)| dr, \end{aligned}$$

where we use (2.9), and $I_r^{t,t'}(x, y) := \nabla v(T-r, X_r^{t'}(x+y)) - \nabla v(T-r, X_r^t(x)) - (\nabla v(T-r, X_r^t(x+y)) - \nabla v(T-r, X_r^t(x)))$, $J_r^t(x, y) := \nabla X_r^t(x+y) - \nabla X_r^t(x)$. Combining all the estimate above, by Grownwall lemma, for every $x, y \in \mathbb{R}^d$, $0 \leq t \leq t' \leq s \leq T$,

$$\lim_{t' \rightarrow t} |\Theta_s^{t,t'}(x, y)| = 0, \text{ a.s..}$$

By (2.9) and Grownwall lemma, it is not difficult to check that,

$$\begin{aligned} |I_r^{t,t'}(x, y)| & \leq CKe^{CKT}(|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}), \\ |J_r^t(x, y)| & \leq CKe^{CKT}|r - t|(|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}), \end{aligned}$$

so

$$\sup_{0 \leq t \leq t' \leq s \leq T} |\Theta_s^{t,t'}(x, y)| \leq CK e^{CKT} (1 + T + |B_{t'} - B_t|) (|y|^{r(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}),$$

hence by the dominated convergence theorem,

$$\boxed{112.1} \quad (3.58) \quad \lim_{t' \rightarrow t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Theta_s^{t,t'}(x, y)|^p |\nabla^2 v(T - s, X_s^t(x))|^p}{|y|^{d+\alpha p}} dx dy = 0.$$

By the same procedure of Step 4, we have,

$$\lim_{t' \rightarrow t} \|\nabla^2 v(T - s, X_s^{t'}(\cdot)) - \nabla^2 v(T - s, X_s^t(\cdot))\|_{L^p} = 0,$$

and by the estimate above for $J_r^t(x, y)$,

$$\boxed{112.2} \quad (3.59) \quad \lim_{t' \rightarrow t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla^2 v(T - s, X_s^{t'}(x)) - \nabla^2 v(T - s, X_s^t(x))|^p |J_s^t(x, y)|^p}{|y|^{d+\alpha p}} dx dy = 0.$$

By (3.57) and Step 3 in the proof of Lemma 3.1,

$$\boxed{112.3} \quad (3.60) \quad \begin{aligned} & \lim_{t' \rightarrow t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\nabla^2 v(T - s, X_s^{t'}(x + y)) - \nabla^2 v(T - s, X_s^{t'}(x))|^p |\Lambda_s^{t,t'}(x)|^p}{|y|^{d+\alpha p}} dx dy \\ & \leq C e^{CKT} [v(T - s)]_{B_{p,p}^{2+\alpha}}^p \lim_{t' \rightarrow t} \sup_{x \in \mathbb{R}^d} |\Lambda_s^{t,t'}(x)|^p = 0. \end{aligned}$$

By the inequality (2.13), and according to (3.53), (3.58), (3.59), (3.60),

$$\boxed{e33aa} \quad (3.61) \quad \lim_{t' \rightarrow t} [\nabla^2 v(T - s, X_s^{t'}(\cdot)) \nabla X_s^{t'}(\cdot) - \nabla^2 v(T - s, X_s^t(\cdot)) \nabla X_s^t(\cdot)]_{B_{p,p}^\alpha} = 0.$$

Applying (2.13) to the first term on the right hand side of (3.51), in the same way as above to estimate the associated terms, and using (3.61), Gronwall lemma and the dominated convergence theorem, we have,

$$\lim_{t' \rightarrow t} [\nabla^2 X_s^{t'} - \nabla^2 X_s^t]_{B_{p,p}^\alpha} = 0,$$

since

$$\sup_{0 \leq t \leq s \leq T} \mathbb{E}([\nabla^2 X_s^t]_{B_{p,p}^\alpha}^p) < \infty,$$

$[\nabla^2 X_s^{t'} - \nabla^2 X_s^t]_{B_{p,p}^\alpha}$ is uniformly integrable, and (3.52) holds.

Step 6: By the approximation procedure above, we know $\hat{v} = \mathcal{J}_\nu(v)$ satisfies that $\hat{v}(t) = \mathbf{P}(g(t))$, where

$$g(t, x) = \mathbb{E}(u_0(X_T^t(x))) + \int_t^T \mathbb{E}(F_v(T - s, X_s^t(x))) ds,$$

and for every $t \in [0, T]$, $g(t) \in B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$,

$$\begin{aligned} \nabla^2 g(t, x) &= \mathbb{E}(\nabla^2 u_0(X_T^t(x))(\nabla X_T^t(x))^2 + \nabla u_0(X_T^t(x))\nabla^2 X_T^t(x)) \\ &+ \int_t^T \mathbb{E}(\nabla^2 F_v(T-s, X_s^t(x))(\nabla X_s^t(x))^2 + \nabla F_v(T-s, X_s^t(x))\nabla^2 X_s^t(x))ds, \end{aligned}$$

based on such expression, by (3.53), (3.52) and by the same methods above, we can show that,

$$\lim_{t' \rightarrow t} [\hat{v}(t') - \hat{v}(t)]_{B_{p,p}^{2+\alpha}} \leq C \lim_{t' \rightarrow t} [g(t') - g(t)]_{B_{p,p}^{2+\alpha}} = 0.$$

Since we have shown that $\hat{v} \in C([0, T]; W^{2,p}(\mathbb{R}^d; \mathbb{R}^d))$, we obtain $\hat{v} \in C([0, T]; B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$ and the proof is finished. \square

r3.1 Remark 3.11. Since for $p > 1$, $r > 1 + \frac{d}{p}$, $v \in C([0, T]; B_{p,p}^r(\mathbb{R}^d))$ implies that $v(t)$ and F_v are Lipschitz continuous functions, there exists a unique solution (X, Y, Z) for the forward-backward SDE (2.5) with coefficients v and initial value $u_0 = v(0)$. If we let $g(t) := Y_{T-t}^{T-t}$, by the approximation procedure in the proof of Proposition 3.10, we know that $\mathcal{J}_\nu(v)(t) = \mathbf{P}(g(t))$.

t1 Theorem 3.12. Suppose $p > 1$, $r > 1 + \frac{d}{p}$ and $u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ satisfying that $\nabla \cdot u_0 = 0$; then there exists a constant T_0 , which depends only on $\|u_0\|_{B_{p,p}^r}$ (in particular, T_0 is independent of the viscosity ν), for which there is a unique fixed point u of the map \mathcal{J}_ν in $\mathcal{B}(u_0, T_0, p, r)$, where $\mathcal{B}(u_0, T_0, p, r)$ is defined in (3.45).

Proof. For each $T > 0$ and $v \in \mathcal{B}(u_0, T, p, r)$ with $\sup_{t \in [0, T]} \|v(t)\|_{B_{p,p}^r} \leq K$, by Proposition 3.10, (3.44) holds with g replaced by $\mathcal{J}_\nu(v)$, i.e.,

$$\sup_{t \in [0, T]} \|\mathcal{J}_\nu(v)(t)\|_{B_{p,p}^r} \leq C(1 + T^{[r]}K^{[r]})e^{CKT}\|u_0\|_{B_{p,p}^r} + CT K^2(1 + T^{[r]+1}K^{[r]+1}).$$

Note that the above bound in the right hand side tends to $C\|u_0\|_{B_{p,p}^r}$ as T tends to 0 and C is independent of K . Therefore we can find constants $K_0 \gg \|u_0\|_{B_{p,p}^r}$ and $0 < T_1 < 1$ which only depend on $\|u_0\|_{B_{p,p}^r}$, such that for every $0 < T \leq T_1$, $v \in \mathcal{B}(u_0, T, p, r)$ with $\sup_{t \in [0, T]} \|v(t)\|_{B_{p,p}^r} \leq K_0$,

$$\sup_{t \in [0, T]} \|\mathcal{J}_\nu(v)\|_{B_{p,p}^r} \leq K_0.$$

Fix such K_0 ; by Proposition 3.10, there is a constant $0 < T_0 \leq T_1$ only depending on $\|u_0\|_{B_{p,p}^r}$, such that for each $v_1, v_2 \in \mathcal{B}(u_0, T_0, p, r)$ with $\|v_m\|_{B_{p,p}^r} \leq K_0$, $m = 1, 2$,

e23a (3.62)
$$\sup_{t \in [0, T_0]} \|\mathcal{J}_\nu(v_1)(t) - \mathcal{J}_\nu(v_2)(t)\|_{B_{p,p}^r} \leq \frac{1}{2} \sup_{t \in [0, T_0]} \|v_1(t) - v_2(t)\|_{B_{p,p}^r},$$

where $r' := \max(r - 1, 1)$. For every $v \in C([0, T]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$, let $\|v\|_{B_{p,p}^r, T} := \sup_{t \in [0, T]} \|v(t)\|_{B_{p,p}^r}$. From the analysis above, we know that \mathcal{J}_ν can be viewed as a map $\mathcal{J}_\nu : \mathcal{B}(u_0, T_0, p, r, K_0) \rightarrow \mathcal{B}(u_0, T_0, p, r, K_0)$, where

$$\mathcal{B}(u_0, T_0, p, r, K_0) := \{v \in \mathcal{B}(u_0, T_0, p, r); \|v\|_{B_{p,p}^r, T_0} \leq K_0\},$$

and \mathcal{J}_ν is a contractive map with the $\|\cdot\|_{B_{p,p}^{r'}, T_0}$ norm.

Now we follow the argument in [19, Theorem 2.1]. Choose $u_1 \in \mathcal{B}(u_0, T_0, p, r, K_0)$ (for example, $u_1(t) := u_0$ for every $t \in [0, T_0]$), and define $u_n := \mathcal{J}_\nu(u_{n-1})$ inductively. Then due to (3.62),

$$\|u_{n+1} - u_n\|_{B_{p,p}^{r'}, T_0} \leq \frac{1}{2} \|u_n - u_{n-1}\|_{B_{p,p}^{r'}, T_0},$$

which implies that $\{u_n\}_{n=1}^\infty$ has a strong limit $u \in C([0, T_0]; B_{p,p}^{r'}(\mathbb{R}^d; \mathbb{R}^d))$ in the $\|\cdot\|_{B_{p,p}^{r'}, T_0}$ norm. Since $\sup_n \|u_n\|_{B_{p,p}^r, T_0} \leq K_0$, as the same procedure in the proof of Proposition 3.10, we have $u \in C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$ and $\|u\|_{B_{p,p}^r, T_0} \leq K_0$. So according to (3.62),

$$\|\mathcal{J}_\nu(u_n) - \mathcal{J}_\nu(u)\|_{B_{p,p}^{r'}, T_0} \leq \frac{1}{2} \|u_n - u\|_{B_{p,p}^{r'}, T_0},$$

which implies that $\mathcal{J}_\nu(u) = u$. The uniqueness of the fixed point u also follows from (3.62). \square

Let $C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ denote the set of vector fields which are one order differentiable with respect to the time variable in $[0, T]$ and twice differentiable with respect to the space variable in \mathbb{R}^d . If a solution u of (1.1) belongs to $C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$, it is a classical solution (differentiable in time and space variables). Now we prove that the fixed point u of \mathcal{J}_ν is the solution of Navier-Stokes equation (1.1),

t2 **Theorem 3.13.** *Suppose $p > 1$, $r > 1 + \frac{d}{p}$, and $u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ satisfying that $\nabla \cdot u_0 = 0$. Then there exists a vector field $u \in C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$ for some constant $T_0 > 0$ which only depends on $\|u_0\|_{B_{p,p}^r}$ and is independent of ν , such that u is the unique strong solution of (1.1) in $C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$. In particular, if $r > 2 + \frac{d}{p}$, u is a classical solution of (1.1).*

Proof. Step 1: Suppose $u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ for general $r > 1 + \frac{d}{p}$; as in the proof of Proposition 3.10, we can obtain a sequence $\{u_{0,n}\} \subseteq \bigcap_{l \geq 1} B_{p,p}^l(\mathbb{R}^d; \mathbb{R}^d)$, such that $\lim_{n \rightarrow \infty} \|u_{0,n} - u_0\|_{B_{p,p}^r} = 0$ and $\nabla \cdot u_{0,n} = 0$. Recall the iteration procedure in the proof of Theorem 3.12; we can find a constant T_1 independent of n and ν , such

that for every n , there exist vectors $\{u_{n,m}\}_{m=1}^\infty \subseteq \bigcap_{l \geq 1} C([0, T_1]; B_{p,p}^l(\mathbb{R}^d; \mathbb{R}^d))$, $u_n \in C([0, T_1]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$, such that

e20aa

$$(3.63) \quad \begin{aligned} u_{n,m}(0) &= u_{0,n}, \quad u_{n,m+1} = \mathcal{J}_\nu(u_{n,m}), \quad \sup_{m,n} \sup_{t \in [0, T_1]} \|u_{n,m}(t)\|_{B_{p,p}^r} < \infty, \\ \lim_{m \rightarrow \infty} \sup_{t \in [0, T_1]} \|u_{n,m}(t) - u_n(t)\|_{B_{p,p}^{r'}} &= 0, \quad \mathcal{J}_\nu(u_n) = u_n, \end{aligned}$$

where $r' := \max(r-1, 1)$. Moreover, let $(X_{n,m}, Y_{n,m}, Z_{n,m})$ be the solution of (2.5) with coefficients $v = u_{n,m}$ and initial condition $u_{0,n}$. We define $g_{n,m}(t) := Y_{T_1-t, n, m}^{T_1-t}$ for $t \in [0, T_1]$; since $u_{n,m}$ is regular enough, $g_{n,m}$ is the unique classical solution of the following PDE,

e23c

$$(3.64) \quad \frac{\partial g_{n,m}}{\partial t} + u_{n,m} \cdot \nabla g_{n,m} = \nu \Delta g_{n,m} + F_{u_{n,m}}, \quad g_{n,m}(0) = u_{0,n}.$$

Therefore it is a strong solution in the following sense, for every $t \in [0, T_1]$,

e22c

$$(3.65) \quad g_{n,m}(t) = e^{t\nu\Delta} u_{0,n} - \int_0^t \left(e^{(t-s)\nu\Delta} (u_{n,m}(s) \cdot \nabla g_{n,m}(s) - F_{u_{n,m}}(s)) \right) ds.$$

Suppose T_1 independent of n, m small enough, by (3.43) and (3.63) we have,

$$\sup_{t \in [0, T_1]} \|g_{n,m+1}(t) - g_{n,m}(t)\|_{B_{p,p}^{r'}} \leq 2 \sup_{t \in [0, T_1]} \|u_{n,m+1}(t) - u_{n,m}(t)\|_{B_{p,p}^{r'}},$$

so by (3.63) and by the same argument in the proof of Proposition 3.10, there is a $g_n \in C([0, T_1]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$, such that,

e22aa

$$(3.66) \quad \lim_{m \rightarrow \infty} \sup_{t \in [0, T_1]} \|g_{n,m}(t) - g_n(t)\|_{B_{p,p}^{r'}} = 0, \quad \sup_n \sup_{t \in [0, T_1]} \|g_n(t)\|_{B_{p,p}^r} < \infty.$$

Letting $m \rightarrow \infty$ in (3.65) we obtain, for every $t \in [0, T_1]$,

e23

$$(3.67) \quad g_n(t) = e^{t\nu\Delta} u_{0,n} - \int_0^t \left(e^{(t-s)\nu\Delta} (u_n(s) \cdot \nabla g_n(s) - F_{u_n}(s)) \right) ds.$$

Step 2: Since $\nabla \cdot u_{n,m}(t) = 0$, by definition (2.4) and standard approximation procedure (see Lemma 2.2), for every t ,

$$\nabla \cdot F_{u_{n,m}}(t) = \nabla \cdot (\nabla N G_{u_{n,m}}(t)) = \Delta N G_{u_{n,m}}(t) = G_{u_{n,m}}(t).$$

Let

$$H_{u_{n,m}, g_{n,m}}(t) := \sum_{i,j=1}^d (\partial_i u_{n,m}^j(t) \partial_j (g_{n,m}^i(t) - u_{n,m}^i(t))).$$

Let $h_{n,m}(t) := \nabla \cdot g_{n,m}(t)$, taking the divergence in (3.64), so for every $t \in [0, T_1]$,

$$\frac{\partial h_{n,m}}{\partial t} + u_{n,m} \cdot \nabla h_{n,m} = \nu \Delta h_{n,m} - H_{u_{n,m}, g_{n,m}}, \quad h_{n,m}(0) = 0.$$

For every $0 \leq t \leq s \leq T_1$, applying Ito's formula to $h_{n,m}(T_1 - s, X_{s,n,m}^t(x))$ and taking the expectation, we get,

$$h_{n,m}(T_1 - t, x) = - \int_t^{T_1} \mathbb{E}(H_{u_{n,m}, g_{n,m}}(T_1 - s, X_{s,n,m}^t(x))) ds,$$

hence for every $0 \leq t \leq T_1$,

$$\|h_{n,m}(t)\|_{L^p} \leq C \int_0^t \|H_{u_{n,m}, g_{n,m}}(s)\|_{L^p} ds,$$

so by (3.63), let $m \rightarrow \infty$,

$$\boxed{\text{e22d}} \quad (3.68) \quad \|h_n(t)\|_{L^p} \leq C \int_0^t \|H_{u_n, g_n}(s)\|_{L^p} ds,$$

where $h_n(t) := \nabla \cdot g_n(t)$, and

$$H_{u_n, g_n}(t) := \sum_{i,j=1}^d (\partial_i u_n^j(t) \partial_j (g_n^i(t) - u_n^i(t))).$$

Since for every $v \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$, the Leray-Hodge projection has the expression $v - \mathbf{P}v = \nabla N(\nabla \cdot v)$, for every $p > 1$ we have,

$$\begin{aligned} \|\nabla(v - \mathbf{P}v)\|_{L^p} &= \|\nabla^2 N(\nabla \cdot v)\|_{L^p} \\ &\leq C \|\Delta N(\nabla \cdot v)\|_{L^p} = C \|\nabla \cdot v\|_{L^p}, \end{aligned}$$

where in the second step we use the elliptic regularity estimate (for example, see [18]) $\|\nabla^2 f\|_{L^p} \leq C \|\Delta f\|_{L^p}$ for every $f \in C_c^\infty(\mathbb{R}^d)$. Note that $\mathbf{P}(g_n(t)) = u_n(t)$ as $\mathbf{P}(g_{n,m}(t)) = u_{n,m+1}(t)$, by the standard approximation argument,

$$\|\nabla(u_n(t) - g_n(t))\|_{L^p} \leq C \|\nabla \cdot g_n(t)\|_{L^p} = C \|h_n(t)\|_{L^p},$$

which implies

$$\boxed{\text{e22}} \quad (3.69) \quad \|H_{u_n, g_n}(t)\|_{L^p} \leq CK \|h_n(t)\|_{L^p},$$

where $K := \sup_n \sup_{t \in [0, T]} (\|\nabla u_n(t)\|_{L^\infty})$. According to (3.68), (3.69) and Grownwall lemma, we derive $\|h_n(t)\|_{L^p} = 0$ for every $t \in [0, T_1]$. So $\nabla \cdot g_n(t) = 0$ and $g_n(t) = \mathbf{P}g_n(t) = u_n(t)$.

Since $\mathcal{J}_\nu(u_n) = u_n$, by (3.43) and (3.63), there is a $0 < T_0 \leq T_1$ independent of ν , n and a vector $u \in C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$, such that,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T_0]} \|u_n(t) - u(t)\|_{B_{p,p}^{r'}} \leq 2 \lim_{n \rightarrow \infty} \|u_{0,n} - u_0\|_{B_{p,p}^{r'}} = 0,$$

so taking the limit $n \rightarrow \infty$ in (3.67) we have, for every $t \in [0, T_0]$,

$$\begin{aligned} u(t) &= e^{t\nu\Delta} u_0 - \int_0^t \left(e^{(t-s)\nu\Delta} (u(s) \cdot \nabla u(s) - F_u(s)) \right) ds \\ &= e^{t\nu\Delta} u_0 - \int_0^t \left(e^{(t-s)\nu\Delta} (\mathbf{P}(u(s) \cdot \nabla u(s))) \right) ds. \end{aligned}$$

Hence $u \in C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$ is the strong solution of (1.1) introduced in [17]. In particular, if $r > 2 + \frac{d}{p}$, by Sobolev embedding theorem, u is a classical solution.

Step 3: Suppose $r > 1 + \frac{d}{p}$ and $u \in C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$ is a strong solution of (1.1). Without loss of generality, we assume T_0 to be small enough. Note that under such regularity condition, the backward SDE (2.5) with coefficients u and initial condition u_0 has a unique solution (X, Y, Z) . Let $g(t) := Y_{T_0-t}^{T_0-t}$ for $t \in [0, T_0]$. By (3.43) and the approximation procedure above, $g \in C([0, T_0]; B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d))$ is the strong solution of following (linear) PDE,

e1a

$$(3.70) \quad \frac{\partial g}{\partial t} + u \cdot \nabla g = \nu \Delta g + F_u, \quad g(0) = u_0.$$

On the other hand, since u is a strong solution of (1.1), u is also a strong solution of (3.70). Due to the uniqueness of the strong solution of linear PDE (3.70) in such function space, we must have $g(t) = u(t)$, so $u = g = \mathcal{J}_\nu(u)$ (see Remark 3.11), hence it is a fixed point of \mathcal{J}_ν in $\mathcal{B}(u_0, T_0, p, r)$ and it is unique according to Theorem 3.12. \square

Remark 3.14. As we will see in Section 5, in order to prove Theorem 3.12 and 3.13, the estimate for $W^{1,p}$ norm of the difference is sufficient. Here we prove the estimate for $B_{p,p}^{1+\alpha}$ norm in Lemma 3.7 and, based on such estimate, we can obtain a more accurate rate for the limit as $\nu \rightarrow 0$ in Section 4.

4 The limit to the Euler equation as $\nu \rightarrow 0$

From the analysis in Section 3, we know that the maximal time interval $[0, T_0]$ for the local existence of a solution for (1.1) is independent of the viscosity ν . Although when $\nu = 0$, the backward SDE in (2.5) makes no sense, the function $g(t)$ is still well defined by (3.19) since X_s^t here is the solution of an ODE. Furthermore, the proof of

Proposition 3.10, Theorem 3.12, 3.13 can still be applied to the case where $\nu = 0$, and for $p > 1$, $r > 1 + \frac{d}{p}$, the local existence theorem in Besov space $B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ for the Euler equation (equation (1.1) with $\nu = 0$) can be derived. In this section, we will study the limit behaviour of the solution of (1.1) as $\nu \rightarrow 0$.

For every $p > 1$, $r > 1 + \frac{d}{p}$, $T > 0$, $u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$, let $\mathcal{B}(u_0, T, p, r)$ be the set defined by (3.45). For any $\nu \geq 0$, $v \in \mathcal{B}(u_0, T, p, r)$, let $\mathcal{J}_\nu : \mathcal{B}(u_0, T, p, r) \rightarrow \mathcal{B}(u_0, T, p, r)$ be the map constructed in Proposition 3.10. By Theorem 3.12, given a $u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ with $\nabla \cdot u_0 = 0$, there is a constant $T_0 > 0$ independent of ν such that for every $\nu \geq 0$, there is a vector u_ν which is a fixed point of \mathcal{J}_ν in the space $\mathcal{B}(u_0, T_0, p, r)$. For not making the notation confusing, we denote u_ν with $\nu = 0$ by u , and the initial point is denoted by u_0 . Let X_ν , X be the solution of first equation in (2.5) with coefficients u_ν and u respectively, and with the same driven Brownian motion B_t .

In this section, we consider $B_{p,p}^{2+\alpha}$ norm for simplicity, the other cases can be shown similarly. We define, $K := \sup_\nu \sup_{t \in [0, T_0]} \|u_\nu(t)\|_{B_{p,p}^{2+\alpha}} < \infty$ and in the proof of the lemmas in this section, the constant C will change in different line, but will not depend on the variable stated in the conclusion of the lemmas. We first show the following estimate:

14.1 **Lemma 4.1.** *Suppose $u_0 \in B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$ for some $d < p < \infty$, $0 < \alpha < 1$, then for every $f_1, f_2 \in W^{1,p}(\mathbb{R}^d)$,*

$$\begin{aligned} & \int_{\mathbb{R}^d} |f_1(X_{s,\nu}^t(x)) - f_2(X_s^t(x))|^p dx \\ & \leq C_1 \|f_1 - f_2\|_{L^p}^p + C_1 e^{C_1 K T_0} \|\nabla f_2\|_{L^p}^p \left(T_0 \sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{L^\infty}^p + (\sqrt{2\nu} |B_s - B_t|)^p \right) \text{ a.s.,} \end{aligned}$$

where C_1 is a positive constant independent of ν , T_0 , K , and f_m .

Proof. The proof is quite similar to the one of Lemma 3.4.

By the approximation argument, it is enough to prove the conclusion for every $f_1, f_2 \in C^1(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$. Since

$$|f_1(X_{s,\nu}^t(x)) - f_2(X_s^t(x))| \leq |f_1(X_{s,\nu}^t(x)) - f_2(X_{s,\nu}^t(x))| + |f_2(X_{s,\nu}^t(x)) - f_2(X_s^t(x))|.$$

By (3.3), for every $0 \leq t \leq s \leq T$,

e25aa (4.1)
$$\int_{\mathbb{R}^d} |f_1(X_{s,\nu}^t(x)) - f_2(X_{s,\nu}^t(x))|^p dx \leq \|f_1 - f_2\|_{L^p}^p, \text{ a.s.,}$$

As in Lemma 3.4, for every $r \in [0, 1]$, we define $X_s^{t,r}(x)$ to be the solution of following SDE,

e25a (4.2)
$$\begin{cases} dX_s^{t,r}(x) = r\sqrt{2\nu} dB_s - u_{r,\nu}(T-s, X_s^{t,r}(x)) ds, \\ X_t^{t,r}(x) = x, \quad 0 \leq t \leq s \leq T_0, \end{cases}$$

where $u_{r,\nu}(t, x) := (1 - r)u(t, x) + ru_\nu(t, x)$. Clearly we have $X_s^{t,r}(x) = X_s^t(x)$ if $r = 0$ and $X_s^{t,x}(x) = X_{s,\nu}^t(x)$ if $r = 1$.

Since $u_\nu(t) \in B_{p,p}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$, $u_{r,\nu}(t) \in C_b^{1,r(p)}(\mathbb{R}^d; \mathbb{R}^d)$, by the argument in [22] there is a version of $X_s^{t,r}(x)$ which is differentiable with r , and $V_s^{t,r}(x) := \frac{d}{dr}(X_s^{t,r}(x))$ satisfies the following SDE,

$$\boxed{\text{e26}} \quad (4.3) \quad \begin{cases} dV_s^{t,r}(x) = \sqrt{2\nu}dB_s - u_{r,\nu}(T-s, X_s^{t,r}(x))V_s^{t,r}(x)ds \\ + (u(T-s, X_s^{t,r}(x)) - u_\nu(T-s, X_s^{t,r}(x)))ds, \\ V_t^{t,r}(x) = 0, \quad 0 \leq t \leq s \leq T_0. \end{cases}$$

Comparing with equation (3.24), the martingale part of (4.3) does not vanish. By Grownwall Lemma, for every $0 \leq t \leq s \leq T_0$, $r \in [0, 1]$ and $x \in \mathbb{R}^d$,

$$\boxed{\text{e26a}} \quad (4.4) \quad |V_s^{t,r}(x)| \leq Ce^{CKT_0} \left(T_0 \sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{L^\infty} + \sqrt{2\nu}|B_s - B_t| \right), \text{ a.s..}$$

Also note that $f_2 \in C^1(\mathbb{R}^d)$; then, following the same procedure in Lemma 3.4 and especially (3.26), we can show that,

$$\begin{aligned} & \int_{\mathbb{R}^d} |f_2(X_{s,\nu}^t(x)) - f_2(X_s^t(x))|^p dx \\ & \leq Ce^{CKT_0} \int_{\mathbb{R}^d} |\nabla f_2(x)|^p dx \left(T_0 \sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{L^\infty}^p + (\sqrt{2\nu}|B_s - B_t|)^p \right) \text{ a.s.,} \end{aligned}$$

together with (4.1), which allows to prove the conclusion. \square

14.2 **Lemma 4.2.** *Suppose u_0 satisfies the same condition as the one in Lemma 4.1. For every $f_1, f_2 \in B_{p,p}^{1+\alpha}(\mathbb{R}^d)$, $0 \leq t \leq s \leq T_0$,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| (f_1(X_{s,\nu}^t(x+y)) - f_2(X_s^t(x+y))) - (f_1(X_{s,\nu}^t(x)) - f_2(X_s^t(x))) \right|^p}{|y|^{d+p\alpha}} dx dy \\ & \leq C_1 e^{C_1 K T_0} [f_1 - f_2]_{B_{p,p}^\alpha}^p \\ & + C_1 T_0^p (1 + K^p) e^{C_1 K T_0} \|f_2\|_{B_{p,p}^{1+\alpha}}^p \sup_{t \in [0, T_0]} (\|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}}^p + (\sqrt{2\nu}|B_t|)^p) \text{ a.s.,} \end{aligned}$$

where C_1 is a positive constant independent of ν , K , T_0 , f_m .

Proof. By the approximation argument, it is sufficient to prove the conclusion for every $f_1, f_2 \in C^2(\mathbb{R}^d) \cap B_{p,p}^{1+\alpha}(\mathbb{R}^d)$. The proof is almost a repetition of the steps of the proof of Lemma 3.5, the only difference is that we need to use the estimate for the solution $V_s^{t,r}(x)$ of (4.3), rather than that of (3.24).

Let $\Gamma_s^{t,r}(x, y) := V_s^{t,r}(x+y) - V_s^{t,r}(x)$. Since the martingale part of $\Gamma_s^{t,r}(x, y)$ vanishes, we can follow the procedure in the proof of Lemma 3.5 step by step; without loss of generality, we assume $T_0 \leq 1$, so by (4.4) we can derive the following estimate similar to (3.31),

$$\begin{aligned} \text{e27} \quad (4.5) \quad & |\Gamma_s^{t,r}(x, y)| \leq CT_0(1+K)e^{CKT_0} \sup_{t \in [0, T_0]} (\|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}} \\ & + \sqrt{2\nu}|B_t|)(|y|^{r(p)}1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) \text{ a.s..} \end{aligned}$$

Hence based on the estimate (4.4), (4.5) and following the same steps of Lemma 3.5, we prove the conclusion. \square

14.3 Lemma 4.3. *Suppose u_0 satisfies the same conditions in Lemma 4.1. Then, for every $f_1, f_2 \in \bigcap B_{p,p}^{2+\alpha}(\mathbb{R}^d)$ and $0 \leq t \leq s \leq T_0$,*

$$\begin{aligned} & \|f_1 \circ X_{s,\nu}^t(\cdot) - f_2 \circ X_s^t(\cdot)\|_{B_{p,p}^{1+\alpha}} \leq C_1 e^{C_1 K T_0} (1 + T_0 K) \|f_1 - f_2\|_{B_{p,p}^{1+\alpha}} \\ & + C_1 T_0 e^{C_1 K T_0} (1 + K^2) \|f_2\|_{B_{p,p}^{2+\alpha}} \sup_{t \in [0, T_0]} (\|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}} + \sqrt{2\nu}|B_t|) \text{ a.s.,} \end{aligned}$$

where C_1 is a positive constant independent of ν, K, T, f_m .

Proof. By the approximation argument, it is sufficient to prove the conclusion for every $f_1, f_2 \in C^2(\mathbb{R}^d) \cap B_{p,p}^{1+\alpha}(\mathbb{R}^d)$. Note that for $\nabla X_{s,\nu}^t(x) - \nabla X_s^t(x)$, the martingale part vanishes, hence based on Lemma 4.1 and 4.2 and repeating the proof of Lemma 3.6, we can prove the conclusion. \square

Now we can show the following result about the limit behaviour of u_ν .

t4.1 Theorem 4.4. *Suppose u_0 satisfies the same condition as the one in Lemma 4.1; then there is a $0 < T_1 \leq T_0$ (independent of ν), such that,*

$$\text{e27a} \quad (4.6) \quad \sup_{t \in [0, T_1]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}} \leq \sqrt{2\nu T_1},$$

and, for every $0 < \beta < \alpha$,

$$\text{e27c} \quad (4.7) \quad \lim_{\nu \rightarrow 0} \sup_{t \in [0, T_1]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{2+\beta}} = 0.$$

Proof. We define,

$$\begin{aligned} \text{e25} \quad (4.8) \quad & g_\nu(T_0 - t, x) := \mathbb{E}(u_0(X_{T_0,\nu}^t(x))) + \int_t^{T_0} \mathbb{E}(F_{u_\nu}(T_0 - s, X_{s,\nu}^t(x))) ds, \\ & g(T_0 - t, x) := \mathbb{E}(u_0(X_{T_0}^t(x))) + \int_t^{T_0} \mathbb{E}(F_u(T_0 - s, X_s^t(x))) ds. \end{aligned}$$

By Lemma 2.4, Lemma 4.3 and Hölder inequality,

$$\begin{aligned}
& \|\mathbb{E}(F_{u_\nu}(T_0 - s, X_{s,\nu}^t(\cdot)) - F_u(T_0 - s, X_s^t(\cdot)))\|_{B_{p,p}^{1+\alpha}}^p \\
& \leq \mathbb{E}(\|F_{u_\nu}(T_0 - s, X_{s,\nu}^t(\cdot)) - F_u(T_0 - s, X_s^t(\cdot))\|_{B_{p,p}^{1+\alpha}}^p) \\
& \leq C e^{CKT_0} (1 + T_0^p K^p) \|F_{u_\nu}(T - s) - F_u(T - s)\|_{B_{p,p}^{1+\alpha}}^p \\
& \quad + CT_0^p e^{CKT_0} (1 + K^{2p}) \|F_u(T - s)\|_{B_{p,p}^{2+\alpha}}^p \left(\sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}}^p + (\sqrt{2\nu T_0})^p \right).
\end{aligned}$$

Hence by Lemma 2.2 and 2.4,

$$\begin{aligned}
& \|\mathbb{E}(F_{u_\nu}(T_0 - s, X_{s,\nu}^t(\cdot)) - F_u(T_0 - s, X_s^t(\cdot)))\|_{B_{p,p}^{1+\alpha}}^p \\
& \leq CK^p e^{CKT_0} (1 + T_0^p K^p + T_0^p K^p (1 + K^{2p})) \left(\sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}}^p + (\sqrt{2\nu T_0})^p \right).
\end{aligned}$$

Analogously,

$$\begin{aligned}
& \|\mathbb{E}(u_0(X_{T_0,\nu}^t(\cdot)) - u_0(X_{T_0}^t(\cdot)))\|_{B_{p,p}^{1+\alpha}}^p \\
& \leq CT_0^p K^p e^{CKT_0} (1 + K^{2p}) \left(\sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}}^p + (\sqrt{2\nu T_0})^p \right).
\end{aligned}$$

Combining this estimate and (4.8), and noting that $u_\nu(t) - u(t) = \mathbf{P}(g_\nu(t) - g(t))$, we have,

$$\sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}} \leq CT_0 e^{CKT_0} (1 + K^3) \left(\sup_{t \in [0, T_0]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}} + \sqrt{2\nu T_0} \right).$$

Choosing $0 < T_1 \leq T_0$, such that $CT_1 e^{CKT_1} (1 + K^3) \leq \frac{1}{2}$, we obtain,

$$\sup_{t \in [0, T_1]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}} \leq \sqrt{2\nu T_1},$$

which implies (4.6).

For every $0 < \beta < \alpha$, due the interpolation inequality in [32, Theorem 2.4.1(a)], we obtain,

$$\|u_\nu(t) - u(t)\|_{B_{p,p}^{2+\beta}} \leq C \|u_\nu(t) - u(t)\|_{B_{p,p}^{1+\alpha}}^\theta \|u_\nu(t) - u(t)\|_{B_{p,p}^{2+\alpha}}^{1-\theta},$$

where $0 < \theta < 1$ is the unique number such that $\theta(1 + \alpha) + (1 - \theta)(2 + \alpha) = 2 + \beta$. (i.e. $\theta = \beta - \alpha$). Since

$$\sup_\nu \|u_\nu(t) - u(t)\|_{B_{p,p}^{2+\alpha}} \leq 2 \sup_\nu \sup_{t \in [0, T_0]} \|u_\nu(t)\|_{B_{p,p}^{2+\alpha}} \leq 2K,$$

and, according to (4.6), we can show (4.7). □

r4.1 **Remark 4.5.** As stated in Remark 3.8, by the same methods above, we can estimate lower and higher order Besov norms. More precisely, if $u_0 \in B_{p,p}^r(\mathbb{R}^d; \mathbb{R}^d)$ for some $p > 1$, $r > 1 + \frac{d}{p}$, there exists a constant $0 < T_1 \leq T_0$ such that,

$$\sup_{t \in [0, T_1]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{\max(r-1, 1)}} \leq \sqrt{2\nu T_1}$$

and, for every $0 < \tilde{r} < r$,

$$\lim_{\nu \rightarrow 0} \sup_{t \in [0, T_1]} \|u_\nu(t) - u(t)\|_{B_{p,p}^{\tilde{r}}} = 0.$$

5 The local existence theorem in $B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d)$

As pointed out in Section 3, if we use the "Lagrangian path" (forward equation) in (2.3), then we are unable to derive similar estimates for $B_{p,q}^r(p \neq q)$ norms. In this section we will adopt a different "Lagrangian path", which is just a translation by a Brownian motion; together with the associated forward-backward stochastic differential system similar to (2.5), we can establish useful estimates for $B_{p,q}^r$ norms, which however depend on the viscosity ν . Therefore they can not be applied to the case of $\nu = 0$, i.e. to the Euler equation.

As in Section 2, by Itô's formula and the theorem of backward SDEs, u is a (regular enough) solution of (1.1) in time interval $[0, T]$, if and only if $(X_s^t(x), Y_s^t(x), Z_s^t(x), u(t, x), p(t, x))$ satisfies the following (forward-) backward stochastic differential system,

$$\text{e28a} \quad (5.1) \quad \begin{cases} dX_s^t(x) = \sqrt{2\nu} dB_s \\ dY_s^t(x) = \sqrt{2\nu} Z_s^t(x) dB_s + u(T-s, X_s^t(x)) \cdot Z_s^t(x) + \nabla p(T-s, X_s^t(x)) ds \\ Y_t^t(x) = u(T-t, x), \quad \Delta p(t, x) = -\sum_{i,j=1}^3 \partial_i u^j(t, x) \partial_j u^i(t, x) \\ X_t^t(x) = x, Y_T^t(x) = u_0(X_T^t(x)). \end{cases}$$

For every $v \in \mathcal{S}(p, q, p', T)$ with $v(0) = u_0$ for some $1 < p < \infty$, $1 \leq q \leq \infty$, $1 < p' < \frac{d}{2}$, we consider the following (forward-) backward SDE,

$$\text{e29} \quad (5.2) \quad \begin{cases} dX_s^t(x) = \sqrt{2\nu} dB_s \\ dY_s^t(x) = \sqrt{2\nu} Z_s^t(x) dB_s + v(T-s, X_s^t(x)) \cdot Z_s^t(x) - F_v(T-s, X_s^t(x)) ds \\ X_t^t(x) = x, Y_T^t(x) = u_0(X_T^t(x)), \end{cases}$$

where the vector F_v is defined by (2.4).

We first cite the following well-known Bismut-Elworthy-Li formula, e.g. see [5], [16],

15.0 **Lemma 5.1.** Let $X_s^t(x) = x + \sqrt{2\nu}(B_s - B_t)$ for every $0 \leq t < s \leq T$. Then for each $f \in C_b(\mathbb{R}^d)$,

$$\text{e45} \quad (5.3) \quad \nabla(\mathbb{E}(f(X_s^t(x)))) = \mathbb{E}(\nabla f(X_s^t(x))) = \frac{1}{\sqrt{2\nu}(s-t)} \mathbb{E}(f(X_s^t(x))(B_s - B_t)).$$

We have the following estimate,

15.1 **Lemma 5.2.** Suppose $v \in \mathcal{S}(p, q, p', T)$ for some $1 < p < \infty$, $1 \leq q \leq \infty$, $1 < p' < \frac{d}{2}$, $0 < T < 1$. Let (X, Y, Z) be the unique solution of (5.2) with coefficients v and initial condition $u_0 = v(0)$, and let $g(t, x) := Y_{T-t}^T(x)$. Then for every $0 < \alpha < 1$, $\max(1, 2 - \alpha) < \beta < 2$, there exists a $0 < T_0 \leq 1$ which only depends on $K_1, K_{2,\beta}, \nu$, such that for every $0 < T < T_0$,

$$\text{e24} \quad (5.4) \quad \begin{aligned} \|g\|_{1,T} &\leq C_1 \|u_0\|_{B_{p,q}^{1+\alpha}} + C_1 K_1 K_{2,\beta} T^{\frac{\beta-1}{\beta}}, \\ \|g\|_{2,\beta,T} &\leq C_1 \|u_0\|_{B_{p,q}^{1+\alpha}} (\nu^{-\frac{1}{2}} + T^{\frac{1}{2}}) + CT^{\frac{2\alpha+3\beta-4}{2\beta}} (K_1^2 + K_{2,\beta}^2), \end{aligned}$$

where $\|g\|_{1,T} := \sup_{t \in [0,T]} \|g(t)\|_{B_{p,q}^{1+\alpha}}$, $\|g\|_{2,\beta,T} := (\int_0^T \|g(t)\|_{B_{p,q}^{2+\alpha}}^\beta dt)^{\frac{1}{\beta}}$, $K_1 := \sup_{t \in [0,T]} \|v(t)\|_{B_{p,q}^{1+\alpha}}$, $K_{2,\beta} := (\int_0^T \|v(t)\|_{B_{p,q}^{2+\alpha}}^\beta dt)^{\frac{1}{\beta}}$, and C_1 is a constant independent of K_1, K_2, ν, T, p' and v .

Proof. Some parts of the proof are inspired by reference [21].

Step 1: Let $K_1(t) := \|v(t)\|_{B_{p,q}^{1+\alpha}}$, $K_2(t) := \|v(t)\|_{B_{p,q}^{2+\alpha}}$, $\|g\|_{3,T} := \sup_{t \in (0,T]} \{t^{\frac{1}{2}} \|g(t)\|_{B_{p,q}^{2+\alpha}}\}$. Since $\|g\|_{2,\beta,T} \leq C \|g\|_{3,T}$, it is sufficient to prove (5.4) for $\|g\|_{3,T}$.

Since $v \in \mathcal{S}(p, q, p', T)$, by [28, Theorem 2.9], we can find a version of $(X_s^t(x), Y_s^t(x), Z_s^t(x))$ which is a.s. differentiable (for any order) with respect to x and, according to [28], we know that for every $0 \leq t \leq T$, $l > 0$,

$$\text{e29a} \quad (5.5) \quad \sum_{k=0}^2 \mathbb{E} \left(\sup_{t \leq s \leq T} (|\nabla^k Y_s^t(x)|^l + |\nabla^k Z_s^t(x)|^l) \right) < \infty.$$

From [28, Lemma 2.5], for $g(t, x) := Y_{T-t}^T(x)$ we have $Y_s^t(x) = g(T-s, X_s^t(x))$, $Z_s^t(x) = \nabla Y_s^t(x) = \nabla g(T-s, X_s^t(x))$ for every $0 \leq t \leq s \leq T$ (note that the $Z_s^t(x)$ in this paper is actually $\frac{Z_s^t(x)}{\sqrt{\nu}}$ in [28]).

Taking the expectation in (5.2), and taking the derivative in x , for every $0 \leq t \leq T$,

$$\begin{aligned} \text{e30} \quad (5.6) \quad &\nabla g(T-t, x) = \mathbb{E}(\nabla g(T-t, X_t^t(x))) \\ &= \mathbb{E}(\nabla u_0(X_T^t(x))) - \int_t^T \mathbb{E}(\nabla(v \nabla g)(T-r, X_r^t(x))) dr + \int_t^T \mathbb{E}(\nabla F_v(T-r, X_r^t(x))) dr \\ &:= I_0^t(x) + \sum_{i=1}^2 \int_t^T I_{r,i}^t(x) dr. \end{aligned}$$

Since $X_s^t(x) = x + B_s - B_t$, it is easy to check that for every $f \in C_c^\infty(\mathbb{R}^d)$, $l > 1$, $y \in \mathbb{R}^d$,

$$\boxed{\text{e35a}} \quad (5.7) \quad \int_{\mathbb{R}^d} f(X_s^t(x)) dx = \int_{\mathbb{R}^d} f(x) dx, \text{ a.s.},$$

$$\boxed{\text{e39}} \quad (5.8) \quad \|f(X_s^t(\cdot + y)) - f(X_s^t(\cdot))\|_{L^l} = \|f(\cdot + y) - f(\cdot)\|_{L^l}, \text{ a.s..}$$

We first consider the case $1 \leq q < \infty$. For every $y \in \mathbb{R}^d$, by (5.8) and Hölder inequality,

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{\|I_0^t(\cdot + y) - I_0^t(\cdot)\|_{L^p}^q}{|y|^{3+q\alpha}} dy \\ & \leq \int_{\mathbb{R}^d} \frac{\left(\int_{\mathbb{R}^d} \mathbb{E}(|\nabla u_0(X_T^t(x + y)) - \nabla u_0(X_T^t(x))|^p) dx \right)^{\frac{q}{p}}}{|y|^{3+q\alpha}} dy \\ & \leq \int_{\mathbb{R}^d} \frac{\|\nabla u_0(\cdot + y) - \nabla u_0(\cdot)\|_{L^p}^q}{|y|^{3+q\alpha}} dy \leq \|u_0\|_{B_{p,q}^{1+\alpha}}^q. \end{aligned}$$

Then we have,

$$\|I_0^t(\cdot)\|_{B_{p,q}^\alpha} \leq C \|u_0\|_{B_{p,q}^{1+\alpha}}.$$

By (5.3),

$$\boxed{\text{e30aa}} \quad (5.9) \quad I_{r,1}^t(x) = -\frac{1}{\sqrt{2\nu}(r-t)} \mathbb{E} \left(((v \nabla g)(T-r, X_r^t(x))) (B_r - B_t) \right).$$

According to (2.9), we have,

$$\begin{aligned} & \int_{\mathbb{R}^d} \mathbb{E} \left(|(v \nabla g)(T-r, X_r^t(x+y)) - (v \nabla g)(T-r, X_r^t(x))|^p \right) dx \\ & \leq CK_1(T-r)^p \|\nabla g(T-r, \cdot + y) - \nabla g(T-r, \cdot)\|_{L^p}^p \\ & \quad + C \|v(T-r, X_r^t(\cdot + y)) - v(T-r, X_r^t(\cdot))\|_{L^\infty}^p \|\nabla g(T-r)\|_{L^p}^p \\ & \leq CK_1(T-r)^p \|\nabla g(T-r, \cdot + y) - \nabla g(T-r, \cdot)\|_{L^p}^p \\ & \quad + CK_1(T-r)^p (|y|^{pr(p)} 1_{\{|y| \leq 1\}} + 1_{\{|y| > 1\}}) \|g(T-r)\|_{W^{1,p}}^p, \end{aligned} \quad \boxed{\text{e30a}} \quad (5.10)$$

and by Hölder inequality,

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{\|I_{r,1}^t(\cdot + y) - I_{r,1}^t(\cdot)\|_{L^p}^q}{|y|^{3+q\alpha}} dy \\ & \leq \int_{\mathbb{R}^d} \frac{|\mathbb{E}(|B_r - B_t|^{\frac{p}{p-1}})|^{\frac{q(p-1)}{p}}}{\nu^{\frac{q}{2}}(r-t)^q |y|^{3+q\alpha}} \cdot \left(\int_{\mathbb{R}^d} \mathbb{E} \left(|(v \nabla g)(T-r, X_r^t(x+y)) - (v \nabla g)(T-r, X_r^t(x))|^p \right) dx \right) dy \\ & \leq \frac{CK_1^q}{(\nu(r-t))^{\frac{q}{2}}} \|g(T-r)\|_{B_{p,q}^{1+\alpha}}^q. \end{aligned} \quad \boxed{\text{e36aa}} \quad (5.11)$$

Then we get,

$$\|I_{r,1}^t(\cdot)\|_{B_{p,q}^\alpha} \leq \frac{CK_1}{\nu^{\frac{1}{2}}(r-t)^{\frac{1}{2}}} \|g(T-r)\|_{B_{p,q}^{1+\alpha}}.$$

Similarly, by (2.6), Hölder inequality and noting that $\|\nabla v(t)\|_{L^\infty} \leq CK_2(t)$,

$$\int_t^T \|I_{r,2}^t(\cdot)\|_{B_{p,q}^\alpha} dr \leq CK_1 \int_0^T K_2(s) ds \leq CK_1 K_{2,\beta} T^{\frac{\beta-1}{\beta}}.$$

As above, taking the expectation of (5.2), by (2.6), we deduce that

$$\|g(T-t)\|_{L^p} \leq C\|u_0\|_{L^p} + K_1 \int_t^T \|g(T-r)\|_{W^{1,p}} dr + K_1 \int_t^T K_2(T-r) dr.$$

Combining all estimates we have established above and (5.6), we obtain

$$\boxed{\text{e38}} \quad (5.12) \quad \|g\|_{1,T} \leq C\|u_0\|_{B_{p,q}^{1+\alpha}} + C(T^{\frac{1}{2}}\nu^{-\frac{1}{2}} + T)K_1\|g\|_{1,T} + CK_1K_{2,\beta}T^{\frac{\beta-1}{\beta}}.$$

Step 2: Taking the derivative with respect to x in (5.6), to obtain

$$\begin{aligned} \nabla^2 g(T-t, x) &= \mathbb{E}(\nabla^2 u_0(X_T^t(x))) - \int_t^T \mathbb{E}(\nabla(v\nabla^2 g)(T-r, X_r^t(x))) dr \\ &\quad - \int_t^T \mathbb{E}((\nabla^2 v \nabla g)(T-r, X_r^t(x))) dr - \int_t^T \mathbb{E}((\nabla v \nabla^2 g)(T-r, X_r^t(x))) dr \\ \boxed{\text{e31aa}} \quad (5.13) \quad &+ \int_t^T \mathbb{E}(\nabla F_v(T-r, X_r^t(x))) dr \\ &:= J_0^t(x) + \sum_{i=1}^4 \int_t^T J_{r,i}^t(x) dr. \end{aligned}$$

According to (5.3),

$$\begin{aligned} J_0^t(x) &= \frac{1}{\sqrt{2\nu}(T-t)} \mathbb{E}(\nabla u_0(X_T^t(x))(B_T - B_t)), \\ J_{r,1}^t(x) &= -\frac{1}{\sqrt{2\nu}(r-t)} \mathbb{E}((v\nabla^2 g)(T-r, X_r^t(x))(B_r - B_t)); \end{aligned}$$

analogously to (5.10), (5.11) and using (2.9) we obtain

$$\begin{aligned} \|J_0^t(\cdot)\|_{B_{p,q}^\alpha} &\leq \frac{C\|u_0\|_{B_{p,q}^{1+\alpha}}}{\nu^{\frac{1}{2}}(T-t)^{\frac{1}{2}}}, \\ \|J_{r,1}^t(\cdot)\|_{B_{p,q}^\alpha} &\leq \frac{CK_1\|g(T-r)\|_{B_{p,q}^{2+\alpha}}}{\sqrt{2\nu}(r-t)} \leq \frac{CK_1\|g\|_{3,T}}{\nu^{\frac{1}{2}}(r-t)^{\frac{1}{2}}(T-r)^{\frac{1}{2}}}, \\ \|J_{r,2}^t(\cdot)\|_{B_{p,q}^\alpha} &\leq C(\|g(T-r)\|_{B_{p,q}^{2+\alpha}}\|v(T-r)\|_{W^{2,p}} + \|\nabla g(T-r)\|_{L^\infty}\|v(T-r)\|_{B_{p,q}^{2+\alpha}}), \\ \|J_{r,3}^t(\cdot)\|_{B_{p,q}^\alpha} &\leq C(\|g(T-r)\|_{B_{p,q}^{2+\alpha}}\|\nabla v(T-r)\|_{L^\infty} + \|g(T-r)\|_{W^{2,p}}\|v(T-r)\|_{B_{p,q}^{2+\alpha}}); \end{aligned}$$

and by the interpolation inequality in [32, Theorem 2.4.1], we conclude that

$$\boxed{\text{e32aa}} \quad (5.14) \quad \begin{aligned} \|\nabla g(T-r)\|_{L^\infty} &\leq C\|g(T-r)\|_{W^{2,p}} \leq C\|g(T-r)\|_{B_{p,q}^{1+\alpha}}^\alpha \|g(T-r)\|_{B_{p,q}^{2+\alpha}}^{1-\alpha}, \\ \|\nabla v(T-r)\|_{L^\infty} &\leq C\|v(T-r)\|_{W^{2,p}} \leq CK_1(T-r)^\alpha K_2(T-r)^{1-\alpha}. \end{aligned}$$

Hence by Hölder inequality, for any $\max(1, 2-\alpha) < \beta < 2$,

$$\begin{aligned} \int_t^T \|J_{r,2}^t(\cdot)\|_{B_{p,q}^\alpha} dr &\leq CK_1^\alpha \|g\|_{3,T} \int_0^T \frac{K_2(s)^{1-\alpha}}{s^{\frac{1}{2}}} ds + C\|g\|_{1,T}^\alpha \|g\|_{3,T}^{1-\alpha} \int_0^T \frac{K_2(s)}{s^{\frac{1-\alpha}{2}}} ds, \\ &\leq CK_1^\alpha K_{2,\beta}^{1-\alpha} \|g\|_{3,T} T^{\frac{2\alpha+\beta-2}{2\beta}} + CK_{2,\beta} \|g\|_{1,T}^{1-\alpha} \|g\|_{3,T}^\alpha T^{\frac{\alpha\beta+\beta-2}{2\beta}}, \end{aligned}$$

and

$$\int_t^T \|J_{r,3}^t(\cdot)\|_{B_{p,q}^\alpha} dr \leq CK_1^\alpha K_{2,\beta}^{1-\alpha} \|g\|_{3,T} T^{\frac{2\alpha+\beta-2}{2\beta}} + CK_{2,\beta} \|g\|_{1,T}^\alpha \|g\|_{3,T}^{1-\alpha} T^{\frac{\alpha\beta+\beta-2}{2\beta}}.$$

Similarly, by (2.6) and (5.14),

$$\int_t^T \|J_{r,4}^t(\cdot)\|_{B_{p,q}^\alpha} dr \leq CK_1^\alpha \int_0^T K_2(s)^{2-\alpha} ds \leq CK_1^\alpha K_{2,\beta}^{2-\alpha} T^{\frac{\alpha+\beta-2}{\beta}}.$$

Since $\int_t^T \frac{1}{(r-t)^{\frac{1}{2}}(T-r)^{\frac{1}{2}}} dr = B(\frac{1}{2}, \frac{1}{2})$, we have,

$$\int_t^T \|J_{r,1}^t(\cdot)\|_{B_{p,q}^\alpha} dr \leq CK_1 \|g\|_{3,T}.$$

Using the inequality $a^\alpha b^{1-\alpha} \leq C(a+b)$ for every $a, b > 0$, and putting all above estimates together into (5.13), we obtain,

$$\begin{aligned} \|\nabla^2 g(T-t)\|_{B_{p,q}^\alpha} &\leq \frac{C\|u_0\|_{B_{p,q}^{1+\alpha}}}{\nu^{\frac{1}{2}}(T-t)^{\frac{1}{2}}} + \frac{CK_1\|g\|_{3,T}}{\nu^{\frac{1}{2}}} \\ &\quad + CT^{\frac{\alpha+\beta-2}{\beta}} ((K_1 + K_{2,\beta})\|g\|_{3,T} + K_{2,\beta}\|g\|_{1,T} + K_1^2 + K_{2,\beta}^2), \end{aligned}$$

where we have used the fact that $T^{a_1} \leq T^{a_2}$ for $0 < a_2 \leq a_1$ as $T \leq 1$. Combining this with (5.12), we have

$$\boxed{\text{e37}} \quad (5.15) \quad \begin{aligned} \|g\|_{3,T} &\leq \sup_{t \in [0,T]} \{(T-t)^{\frac{1}{2}}\|g(T-t)\|_{B_{p,q}^{2+\alpha}}\} \leq C\|u_0\|_{B_{p,q}^{1+\alpha}} (\nu^{-\frac{1}{2}} + T^{\frac{1}{2}}) + CK_1 T^{\frac{1}{2}} \nu^{-\frac{1}{2}} (\|g\|_{1,T} \\ &\quad + \|g\|_{3,T}) + CT^{\frac{2\alpha+3\beta-4}{2\beta}} ((K_1 + K_{2,\beta})\|g\|_{3,T} + (K_1 + K_{2,\beta})\|g\|_{1,T} + K_1^2 + K_{2,\beta}^2). \end{aligned}$$

From (5.12), (5.15), if we take $0 < T_0 < 1$ which only depends on $K_1, K_{2,\beta}, \nu$, such that

$$C(T_0^{\frac{1}{2}}\nu^{-\frac{1}{2}} + T_0)K_1 \leq \frac{1}{4}, \quad CK_1T_0^{\frac{1}{2}}\nu^{-\frac{1}{2}} \leq \frac{1}{4}, \quad CT_0^{\frac{2\alpha+3\beta-4}{2\beta}}(K_1 + K_{2,\beta}) \leq \frac{1}{4},$$

then estimate (5.4) holds.

If $q = \infty$, by the same argument as above, we may still prove (5.4). □

We also have the following difference estimate.

15.1a **Lemma 5.3.** *Suppose that $v_m \in \mathcal{S}(p, q, p', T)$, $m = 1, 2$, for some $d < p < \infty$, $1 \leq q \leq \infty$, $1 < p' < \frac{d}{2}$, $0 < T < 1$, let (X_m, Y_m, Z_m) be the unique solution of (2.5) with coefficients v_m and initial condition $u_{0,m} := v_m(0)$, and let $g_m(t, x) := Y_{T-t,m}^{T-t}(x)$. Then for every $0 < \alpha < 1$, $\max(1, 2 - \alpha) < \beta < 2$, there is a constant $0 < T_0 \leq 1$ which only depends on $K_1, K_{2,\beta}, \nu$, such that for every $0 < T \leq T_0$,*

e38a (5.16)
$$\begin{aligned} & \sup_{t \in [0, T]} \|g_1(t) - g_2(t)\|_{W^{1,p}} \leq C_1 \|u_{0,1} - u_{0,2}\|_{W^{1,p}} \\ & + C_1(1 + K_1^2 + K_{2,\beta}^2)T^{\frac{\beta-1}{\beta}}(1 + \nu^{-1}) \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}, \end{aligned}$$

where $K_1 := \sup_{m=1,2} \sup_{t \in [0, T]} \|v_m(t)\|_{B_{p,q}^{1+\alpha}}$, $K_{2,\beta} := \sup_{m=1,2} \left(\int_0^T \|v_m(t)\|_{B_{p,q}^{2+\alpha}}^\beta dt \right)^{\frac{1}{\beta}}$, C_1 is a constant independent of $K_1, K_{2,\beta}, \nu, v_m, p'$ and T .

Proof. Note that for different v_1, v_2 , the forward equation in (5.2) is the same, and for every $f_1, f_2 \in L^p(\mathbb{R}^d)$ we have $\|f_1(X_s^t(\cdot)) - f_2(X_s^t(\cdot))\|_{L^p} = \|f_1 - f_2\|_{L^p}$. As in (5.6), we define,

$$\nabla g_m(T - t, x) := I_{0,m}^t(x) + \sum_{i=1}^2 \int_t^T I_{r,i,m}^t(x) dr, \quad m = 1, 2.$$

It is clear that

$$\|I_{0,1}^t(\cdot) - I_{0,2}^t(\cdot)\|_{L^p} \leq C \|u_{0,1} - u_{0,2}\|_{W^{1,p}}.$$

By (5.3),

$$I_{r,1,m}^t(x) = -\frac{1}{\sqrt{2\nu}(r-t)} \mathbb{E} \left(((v_m \nabla g_m)(T-r, X_r^t(x))) (B_r - B_t) \right),$$

and using Hölder inequality as in (5.11), it is not difficult to show that

$$\begin{aligned} \|I_{r,1,1}^t(\cdot) - I_{r,1,2}^t(\cdot)\|_{L^p} & \leq \frac{C}{\nu^{\frac{1}{2}}(r-t)^{\frac{1}{2}}} (\|v_1(T-r) - v_2(T-r)\|_{L^\infty} \|g_1(T-r)\|_{W^{1,p}} \\ & + \|g_1(T-r) - g_2(T-r)\|_{W^{1,p}} \|v_2(T-r)\|_{L^\infty}). \end{aligned}$$

By Lemma 2.4,

$$\|I_{r,2,1}^t(\cdot) - I_{r,2,2}^t(\cdot)\|_{L^p} \leq C \sup_{m=1,2} \|v_m(T-r)\|_{W^{2,p}} \|v_1(T-r) - v_2(T-r)\|_{W^{1,p}}$$

Noticing that $\|v_m(t)\|_{L^\infty} \leq C\|v_m(t)\|_{W^{1,p}}$, combining all the estimates together and using (5.4). Hence there is some $T_1 > 0$, if $0 < T \leq T_1$, then

$$\begin{aligned} \text{e37a} \quad (5.17) \quad & \sup_{t \in [0, T]} \|\nabla g_1(t) - \nabla g_2(t)\|_{L^p} \leq C\|u_{0,1} - u_{0,2}\|_{W^{1,p}} + CT^{\frac{1}{2}}\nu^{-\frac{1}{2}} \left(K_1 \sup_{t \in [0, T]} \|g_1(t) - g_2(t)\|_{W^{1,p}} \right. \\ & + \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{W^{1,p}} \sup_{t \in [0, T]} \|g_1(t)\|_{W^{1,p}} \Big) + CT^{\frac{\beta-1}{\beta}} K_{2,\beta} \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{W^{1,p}} \\ & \leq C\|u_{0,1} - u_{0,2}\|_{W^{1,p}} + CK_1 T^{\frac{1}{2}}\nu^{-\frac{1}{2}} \sup_{t \in [0, T]} \|g_1(t) - g_2(t)\|_{W^{1,p}} \\ & + C(1 + K_1^2 + K_{2,\beta}^2) T^{\frac{\beta-1}{\beta}} (1 + \nu^{-1}) \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}. \end{aligned}$$

Note that

$$g_m(T-t, x) = \mathbb{E}(u_{0,m}(X_T^t(x))) - \int_t^T \mathbb{E}((v_m \nabla g_m)(T-r, X_r^t(x))) dr + \int_t^T \mathbb{E}(F_{v_m}(T-r, X_r^t(x))) dr,$$

and, by the same procedure as above, for every $0 < T < T_1$,

$$\begin{aligned} \text{e37aa} \quad (5.18) \quad & \sup_{t \in [0, T]} \|g_1(t) - g_2(t)\|_{L^p} \leq C\|u_{0,1} - u_{0,2}\|_{L^p} + CT K_1 \sup_{t \in [0, T]} \|g_1(t) - g_2(t)\|_{W^{1,p}} \\ & + C(1 + K_1^2 + K_{2,\beta}^2) T^{\frac{\beta-1}{\beta}} (1 + \nu^{-1}) \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|_{W^{1,p}}. \end{aligned}$$

Based on (5.17), (5.18), if we take a T_0 which only depends on $K_1, K_{2,\beta}, \nu$, such that,

$$CK_1 T_0^{\frac{1}{2}} \nu^{-\frac{1}{2}} \leq \frac{1}{4}, \quad CK_1 T_0 \leq \frac{1}{4},$$

then the conclusion (5.16) holds. \square

For every $v \in \mathcal{S}(p, q, p', T)$ with some $d < p < \infty$, $1 \leq q \leq \infty$, $0 < T < 1$, $1 < p' < \frac{d}{2}$, $0 < \alpha < 1$, $\max(1, 2-\alpha) < \beta < 2$, we can define a map $\mathcal{J}'_\nu : \mathcal{S}(p, q, p', T) \rightarrow C([0, T]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T]; B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$ by $\mathcal{J}'_\nu(v)(t) := \mathbf{P}(Y_{T-t}^{T-t}(\cdot))$, where Y_s^t is the solution of (5.2) with coefficients v and initial condition $u_0 = v(0)$, and \mathbf{P} is the Leray-Hodge projection operator.

Analogously to Proposition 3.10 and Theorem 3.12, we can show the following results about the extension of \mathcal{J}'_ν and its fixed point.

t5.1 **Theorem 5.4.** *Let $0 < \alpha < 1$, $d < p < \infty$, $1 \leq q < \infty$, $\max(1, 2 - \alpha) < \beta < 2$. Suppose $u_0 \in B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$ satisfies $\nabla \cdot u_0 = 0$. Then there exist $K_0 > 0$ and $0 < T_0 \leq 1$ which only depend on $\|u_0\|_{B_{p,q}^{1+\alpha}}$, ν , such that \mathcal{J}'_ν can be extended to be a map $\mathcal{J}'_\nu : \mathcal{B}(u_0, T_0, p, q, \alpha, \beta, K_0) \rightarrow \mathcal{B}(u_0, T_0, p, q, \alpha, \beta, K_0)$, where*

e43aa (5.19)

$$\mathcal{B}(u_0, T_0, p, q, \alpha, \beta, K_0) := \left\{ v \in C([0, T_0]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d)); \right. \\ \left. v(0, x) = u_0(x), \quad \|v\|_{1, T_0} \leq K_0, \quad \|v\|_{2, T_0, \beta} \leq K_0, \quad \nabla \cdot v(t) = 0, \quad \forall t \in [0, T] \right\},$$

where $\|v\|_{1, T_0} := \sup_{t \in [0, T_0]} \|v(t)\|_{B_{p,q}^{1+\alpha}}$ and $\|v\|_{2, T_0, \beta} := \left(\int_0^{T_0} \|v(t)\|_{B_{p,q}^{2+\alpha}}^\beta dt \right)^{\frac{1}{\beta}}$. Moreover, there exists a constant $0 < T_1 < T_0$, which only depends on $\|u_0\|_{B_{p,q}^{1+\alpha}}$ and the viscosity constant ν , such that there is a unique fixed point u for the map \mathcal{J}'_ν in $\mathcal{B}(u_0, T_1, p, q, \alpha, \beta, K_0)$.

Proof. Using the same procedure of the proof of Proposition 3.10, for every $T > 0$, $K > 0$, and $v \in \mathcal{B}(u_0, T, p, q, \alpha, \beta, K)$, there exists a sequence $\{v_n\}_{n=1}^\infty \subseteq \mathcal{S}(p, q, p', T)$ for some $1 < p' < \frac{d}{2}$, such that

e36a (5.20)

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|v_n(t) - v(t)\|_{W^{1,p}} = 0, \\ \sup_n \sup_{t \in [0, T]} \|v_n(t)\|_{B_{p,q}^{1+\alpha}} < \infty, \quad \sup_n \int_0^T \|v_n(t)\|_{B_{p,q}^{2+\alpha}}^\beta dt < \infty.$$

By (5.20) and Lemma 5.2, we can find $K_0 \gg \|u_0\|_{B_{p,q}^{1+\alpha}}$, $0 < T_0 \leq 1$ which only depend on $\|u_0\|_{B_{p,q}^{1+\alpha}}$ and ν , such that for every $v \in \mathcal{B}(u_0, T_0, p, q, \alpha, \beta, K_0)$,

e38aa (5.21)

$$\sup_n \|\mathcal{J}'_\nu(v_n)\|_{1, T_0} \leq K_0, \quad \sup_n \|\mathcal{J}'_\nu(v_n)\|_{2, T_0, \beta} \leq K_0.$$

According to Lemma 5.3, we know that $\{I(v_n)\}_{n=1}^\infty$ is a Cauchy sequence in $C([0, T_0]; W^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$, so it has a limit $\tilde{v} \in C([0, T_0]; W^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$. In particular, such limit \tilde{v} is independent of the choice of sequence $\{v_n\}$, we define $\mathcal{J}'_\nu(v) := \tilde{v}$.

By (5.21), as the same procedure in the proof of Proposition 3.10, we can show $\tilde{v} \in C([0, T_0]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$. In particular, in order to prove the associated estimate (3.53) for $B_{p,q}^\alpha$ norm, $C_c^\infty(\mathbb{R}^d)$ need to be dense in $B_{p,q}^\alpha(\mathbb{R}^d)$, so the case $q = \infty$ can not be included.

Based on (5.20), replacing $B_{p,p}^{r'}$ norm in (3.62) by $W^{1,p}$ norm, and repeating the proof of Theorem 3.12, we can show that there is a constant $0 < T_1 < T_0$ which only depends on $\|u_0\|_{B_{p,q}^{1+\alpha}}$ and ν , such that there exists a unique fixed point for \mathcal{J}'_ν in $\mathcal{B}(u_0, T_1, p, q, \alpha, \beta, K_0)$. \square

Remark 5.5. In contrast with Theorem 3.12, the local existence time T_0 for the fixed point \mathcal{J}'_ν depends on the viscosity constant ν , which is due to the dependence of ν in the estimate (5.4).

Repeating the proof of Theorem 3.13, we can also show that the fixed point u is a solution of (1.1).

t5.2 Theorem 5.6. Let $0 < \alpha < 1$, $d < p < \infty$, $1 \leq q < \infty$, $\max(1, 2 - \alpha) < \beta < 2$. Suppose that $u_0 \in B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)$ which satisfies $\nabla \cdot u_0 = 0$. Then we can find a constant $T_0 > 0$ which only depends $\|u_0\|_{B_{p,q}^{1+\alpha}}$ and ν , such that there exists a vector field $u \in C([0, T_0]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$ which is the unique strong solution of (1.1) in the space $u \in C([0, T_0]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$.

Remark 5.7. Note that in Theorem 5.6 the uniqueness of solution needs to hold in a subspace of $C([0, T]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$, i.e., $C([0, T]; B_{p,q}^{1+\alpha}(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T]; B_{p,q}^{2+\alpha}(\mathbb{R}^d; \mathbb{R}^d))$, since we have to control the norm $\|\nabla v(t)\|_{L^\infty}$ in the iteration procedure.

Also note that for $p > 1$, $r > 1 + \frac{d}{p}$, $\|\nabla v(t)\|_{L^\infty} \leq C\|v(t)\|_{B_{p,q}^r}$, in the similar way as above we can show the following local existence of a unique solution of (1.1) in lower and higher order Besov space.

Suppose $1 < p < \infty$, $1 \leq q < \infty$, $r > \max(1, \frac{d}{p})$, $u_0 \in B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d)$ satisfying that $\nabla \cdot u_0 = 0$. Then we can find a constant $T_0 > 0$ which only depends $\|u_0\|_{B_{p,q}^r}$ and ν , such that there exists a vector field $u \in C([0, T_0]; B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{r+1}(\mathbb{R}^d; \mathbb{R}^d))$ for some $1 < \beta < 2$, which is the unique strong solution for (1.1) in the space $u \in C([0, T_0]; B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{r+1}(\mathbb{R}^d; \mathbb{R}^d))$. Moreover, if $r > 1 + \frac{d}{p}$, the local unique existence of strong solution for (1.1) holds for $u \in C([0, T_0]; B_{p,q}^r(\mathbb{R}^d; \mathbb{R}^d))$.

Remark 5.8. Tracking the proof of Proposition 3.10, we only need the condition that $C_c^\infty(\mathbb{R}^d)$ is dense in $B_{p,q}^\alpha(\mathbb{R}^d)$ to show $\mathcal{J}'_\nu(v)$ is continuous with the time parameter under $B_{p,q}^{1+\alpha}$ norm. Hence if we consider the $B_{p,\infty}^r$ norm for $p > 1$, $r > \max(1, \frac{d}{p})$, the local existence result can be derived with the function space $C([0, T_0]; B_{p,\infty}^r(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,q}^{r+1}(\mathbb{R}^d; \mathbb{R}^d))$ replaced by $L^\infty([0, T_0]; B_{p,\infty}^r(\mathbb{R}^d; \mathbb{R}^d)) \cap C([0, T_0]; B_{p,\infty}^s(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\beta([0, T_0]; B_{p,\infty}^{r+1}(\mathbb{R}^d; \mathbb{R}^d))$ with any $0 < s < r$ and some $1 < \beta < 2$.

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